Existence and Uniqueness Theory for MHD Systems

by

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Contents

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Declarations

All of the work in this thesis has been done in collaboration with my two supervisors James Robinson and Jose Rodrigo, and much of my thesis has already appeared in papers in journals as follows:

- Section 3.1, as well as parts of Chapter 2, formed part (though not all) of McCormick, Robinson & Rodrigo (2013).
- Section 3.2 and Chapter 4 are to appear in McCormick, Robinson & Rodrigo (2014).
- Chapters 5 and 6 have been published in Fefferman, McCormick, Robinson & Rodrigo (2014). This is joint work with Charles Fefferman (Princeton).

Furthermore, Chapter 7 is joint work with Jean-Yves Chemin (Paris 6), which has yet to be submitted for publication.

Except as noted above, I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy, and has not been submitted to any other university or for any other degree.

Abstract

This thesis establishes the existence and uniqueness of solutions to certain systems of equations connected to magnetohydrodynamics (MHD). The models have potential applications to the method of magnetic relaxation introduced by Moffatt (*J. Fluid. Mech.* **159**, 359–378, 1985) to construct stationary Euler flows with non-trivial topology.

Firstly, we prove existence, uniqueness and regularity of weak solutions of a coupled parabolic-elliptic model in 2D, and existence of weak solutions in 3D; we consider the standard equations of MHD with the advective terms removed from the velocity equation. Despite the apparent simplicity of the model, the proof in 2D requires results that are at the limit of what is available, including elliptic regularity in L^1 and a strengthened form of the Ladyzhenskaya inequality

$$\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2} \|\nabla f\|_{L^2}^{1/2}.$$

Secondly, we establish the local-in-time existence and uniqueness of strong solutions in H^s for $s > n/2$ to the viscous, non-resistive MHD equations in $\mathbb{R}^n$, $n = 2, 3$, as well as for a related model where the advection terms are removed from the velocity equation (the above parabolic-elliptic system with zero resistivity). The uniform bounds required for proving existence are established by means of a new estimate, which is a partial generalisation of the commutator estimate of Kato & Ponce (*Comm. Pure Appl. Math.* **41**(7), 891–907, 1988).

Finally, we generalise the results of the previous chapter to prove the local-in-time existence of strong solutions in the Besov space $B^{n/2}_{2,1}(\mathbb{R}^n)$ to the viscous, non-resistive MHD equations in $\mathbb{R}^n$.

Chapter 1

Introduction and Outline

1.1 Introduction

This thesis is concerned with the existence and uniqueness of solutions to various systems of partial differential equations (PDEs) connected to magnetohydrodynamics (MHD), which govern the evolution of a fluid which has an associated magnetic field (e.g. molten iron in the earth's core, or a plasma in a tokamak). As our fundamental model we consider the standard equations of MHD for a velocity field $\boldsymbol{u}$, a magnetic field $\boldsymbol{B}$ and a pressure field p, as follows:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \tag{1.1a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - \eta \Delta \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \tag{1.1b}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0. \tag{1.1c}$$

Here $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$ is the total pressure, $\nu \geq 0$ is the coefficient of viscosity, and $\eta \geq 0$ is the coefficient of magnetic resistivity. The MHD equations are generally derived by coupling the Navier–Stokes equations or the Euler equations, for the velocity field of a fluid, to Maxwell's equations, governing the electric and magnetic fields (see, e.g., Duvaut & Lions (1972)).

Existence and uniqueness theory for MHD is closely connected to the existence and uniqueness theory for the fundamental models of fluid mechanics, the Navier–Stokes and Euler equations:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\nu \geq 0$ is again the coefficient of viscosity; the system with $\nu > 0$ is the Navier–Stokes equations, while the system with $\nu = 0$ is the Euler equations. Both systems have been extensively studied, and global-in-time existence and uniqueness of solutions of either system in 2D is known. However, in 3D, proving global-in-time existence and uniqueness of solutions remains one of the biggest open problems in mathematics.

There are four fundamental results on existence and uniqueness of solutions to the Navier–Stokes equations in 3D, which are outlined below; further details can be found in the review articles of Galdi (2000), Robinson (2006) and Doering (2009).

- Global-in-time existence (but not uniqueness) of *weak* (L^2-valued) solutions to the Navier–Stokes equations was proved by Leray (1934) and Hopf (1951); modern accounts of this result can be found in Constantin & Foias (1988), Doering & Gibbon (1995), and Temam (2001).

- Local-in-time existence of solutions, and global-in-time existence of solutions for *small initial data*, has been proved in a variety of "critical" spaces: noting that the Navier–Stokes equations are invariant under the rescaling

$$\boldsymbol{u}(x,t) \mapsto \boldsymbol{u}_\lambda(x,t) := \lambda \boldsymbol{u}(\lambda x, \lambda^2 t), \qquad p(x,t) \mapsto p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t),$$

  a space X is critical if $\|\boldsymbol{u}_\lambda\|_X = \|\boldsymbol{u}\|_X$ for all $\lambda > 0$. There have been a number of such results over the years (see Lemarié-Rieusset (2002) for an extensive summary), including the result of Kato & Fujita (1962) in $H^{1/2}$ and the result of Kato (1984) in L^3, but the result of Koch & Tataru (2001) in BMO^{-1} is generally considered definitive.

- The "size" of the set of singular points of the Navier–Stokes equations cannot be too big. In particular, Caffarelli, Kohn & Nirenberg (1982) proved that the set of singular points can have Hausdorff dimension at most 1, improving on an earlier result of Scheffer (1976).

- A variety of *conditional regularity* results have been proved over the years, the most important being due to Serrin (1962), Escauriaza, Seregin & Šverák (2003) and others: if a solution $\boldsymbol{u}$ satisfies

$$\int_0^T \|\boldsymbol{u}(\tau)\|_{L^s}^r < \infty$$

  for $2/r + 3/s = 1$ (i.e. $\boldsymbol{u} \in L^r(0,T; L^s(\Omega))$), then the solution can be continued beyond time T.

For the 3D Euler equations, fewer results are known, but the most important is the conditional regularity result of Beale, Kato & Majda (1984), which states that if

$$\int_0^T \|\nabla \times \boldsymbol{u}(\tau)\|_{L^\infty} < \infty,$$

then the solution can be continued beyond time T.

Turning to the theory of existence and uniqueness of solutions of the system (1.1), we divide the known results into four cases, depending on whether each of ν and η are positive or zero.

- In the viscous, resistive case where $\nu, \eta > 0$, in 2D one has global existence and uniqueness of weak solutions, and in 3D one has local existence of weak solutions, much like the Navier–Stokes equations; these results go back to Duvaut & Lions (1972) and Sermange & Temam (1983).

- By contrast, in the non-resistive case with $\nu > 0$ but $\eta = 0$ (that is, diffusion for $\boldsymbol{u}$ but *not* for $\boldsymbol{B}$), Jiu & Niu (2006) established local existence of solutions in 2D for initial data in H^s, but only for integer $s \geq 3$. They also proved a conditional regularity result in 2D: the solution to (1.1) can be extended beyond time T if $\boldsymbol{B} \in L^p(0,T; W^{2,q}(\mathbb{R}^2))$, for $\frac{2}{p} + \frac{1}{q} \leq 2$, and $1 \leq p \leq \frac{4}{3}$, $2 < q \leq \infty$. This was generalised by Zhou & Fan (2011), who showed that $\nabla \boldsymbol{B} \in L^1(0,T; \mathrm{BMO}(\mathbb{R}^2))$ suffices. In 3D, Fan & Ozawa (2009) established a similar conditional regularity result, showing that the solution can be extended beyond time T if $\nabla \boldsymbol{u} \in L^1(0,T; L^\infty(\mathbb{R}^3))$.

- In the inviscid case with $\eta > 0$ but $\nu = 0$ (that is, diffusion for $\boldsymbol{B}$ but *not* for $\boldsymbol{u}$), Kozono (1989) proved *global* existence of weak solutions in 2D for divergence-free initial data in L^2; while in 3D, Fan & Ozawa (2009) showed that, again, the solution can be extended beyond time T if $\nabla \boldsymbol{u} \in L^1(0,T; L^\infty(\mathbb{R}^3))$.

- Finally, in the fully ideal case of $\nu = \eta = 0$, with no diffusion in either equation, Schmidt (1988) and Secchi (1993) established local existence of strong solutions when the initial data is in H^s for integer $s > 1 + n/2$, while Miao & Yuan (2006) established local existence when the initial data is in $B^{n/p+1}_{p,1}$ for $1 < p < \infty$. Furthermore, Caflisch, Klapper & Steele (1997) proved a conditional regularity result for fully ideal MHD which corresponds to the conditional regularity result for Euler due to Beale et al. (1984): namely, if

$$\int_0^T \left(\|\nabla \times \boldsymbol{u}(\tau)\|_{L^\infty} + \|\nabla \times \boldsymbol{B}(\tau)\|_{L^\infty}\right) \mathrm{d}\tau < \infty,$$

then the solution can be continued beyond time T.

The system (1.1) is connected with the method of *magnetic relaxation*, an idea discussed by Moffatt (1985) to construct stationary Euler flows with non-trivial topology. When $\eta = 0$, formally we obtain the standard energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{B}\|_{L^2}^2\right) + \nu\|\nabla\boldsymbol{u}\|_{L^2}^2 = 0;$$

therefore, as long as $\boldsymbol{u}$ is not identically zero, the energy should decay. Thus, the magnetic forces on a viscous non-resistive plasma should come to equilibrium, so that the fluid velocity $\boldsymbol{u}$ tends to zero. We should be left with a steady magnetic field $\boldsymbol{B}$ that satisfies $(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla p_* = 0$, which up to a change of sign for the pressure are the stationary Euler equations.

While this is a useful heuristic argument there is as yet no rigorous proof that the method should yield a stationary Euler flow, not least because there is no global existence result for the system (1.1) with $\eta = 0$, even in 2D. Nonetheless, Núñez (2007) proved that $\|\boldsymbol{u}(t)\|_{L^2} \to 0$ as $t \to \infty$, if we assume a smooth solution to (1.1) exists for all time, and that the solution satisfies $\|\boldsymbol{B}(t)\|_{L^\infty} \le M$ for all t. We should note that Enciso & Peralta-Salas (2012) proved the existence of a stationary Euler flow, albeit with infinite energy, with stream or vortex lines of prescribed link type; but whether such flows arise as limits of system (1.1) (with $\eta = 0$) is still very much open.

However, since the dynamical model used to obtain that steady state is not particularly important, it might prove fruitful to consider an alternative model for magnetic relaxation. In a talk given at the University of Warwick, Moffatt (2009) argued that dropping the acceleration terms from the $\boldsymbol{u}$ equation and working with a "Stokes" model, such as

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.2a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.2b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.2c}$$

might prove more mathematically amenable. Again, however, even with $\eta > 0$ there is no known existence theory for system (1.2).

As a first step towards a rigorous theory of magnetic relaxation for the model (1.2), in this thesis we prove some existence and uniqueness results for (1.2), both in the case $\eta > 0$ and the case $\eta = 0$. We also extend the known theory for (1.1) in the case $\eta = 0$ to prove local-in-time existence of solutions in Sobolev spaces H^s for

$s > n/2$, and the Besov space $B^{n/2}_{2,1}$. (Throughout this thesis, we only consider the case $\nu > 0$.)

1.2 Outline of the Thesis

We now outline the plan for the remainder of the thesis.

- In Chapter 2, we give a brief introduction to the various function spaces in which we will work:
  - Lebesgue (L^p), weak Lebesgue ($L^{p,\infty}$) and Lorentz ($L^{p,q}$) spaces;
  - Sobolev spaces, defined in terms of weak derivatives ($W^{k,p}$) and in terms of Fourier transforms (H^s and $\dot{H}^s$);
  - functions of bounded mean oscillation (BMO);
  - Besov spaces, both homogeneous ($\dot{B}^s_{p,q}$) and inhomogeneous ($B^s_{p,q}$).
- In Chapter 3, we give two proofs of a generalisation of Ladyzhenskaya's inequality involving the weak Lebesgue space $L^{2,\infty}$:

  $$\|f\|_{L^4} \le c\|f\|^{1/2}_{L^{2,\infty}} \|\nabla f\|^{1/2}_{L^2}. \tag{1.3}$$

  The two proofs, using Fourier transforms and interpolation spaces respectively, yield a family of more general results, including a generalisation of the Gagliardo–Nirenberg interpolation inequality.
- In Chapter 4, we consider the Stokes-MHD system (1.2) with $\nu, \eta > 0$; that is,

  $$-\nu\Delta \boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.2a}$$

  $$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.2b}$$

  $$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0. \tag{1.2c}$$

  We prove global existence and uniqueness of (L^2-valued) weak solutions to (1.2) in a bounded domain $\Omega \subset \mathbb{R}^2$ and on the whole of $\mathbb{R}^2$ (see Section 4.2), and we prove global existence (but *not* uniqueness) of weak solutions in a bounded domain $\Omega \subset \mathbb{R}^3$ and on the whole of $\mathbb{R}^3$ (see Section 4.4).

The key to the proof comes from elliptic regularity for the Stokes equations

$$-\nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\boldsymbol{f},$$
$$\nabla\cdot\boldsymbol{u} = 0,$$

in 2D. We show that $\boldsymbol{u} \in L^{2,\infty}$ whenever $\boldsymbol{f} \in L^1$ (see Section 4.1), and then apply the generalised Ladyzhenskaya inequality (1.3).

- In Chapter 5 we prove a new commutator estimate involving the fractional derivative operator Λ^s defined by $\mathscr{F}[\Lambda^s f](\xi) = |\xi|^s \hat{f}(\xi)$. This new estimate is a partial generalisation of the commutator estimate of Kato & Ponce (1988).

  In Section 5.1 we prove that

  $$\|\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] - (\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B})\|_{L^2} \le c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s} \tag{1.4}$$

  for any $s > n/2$.

  One can prove (see Appendix A) that

  $$\begin{aligned}\|\Lambda^{n/2}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] &- (\boldsymbol{u}\cdot\nabla)(\Lambda^{n/2}\boldsymbol{B})\|_{L^2} \\ &\le c(\|\nabla\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\boldsymbol{B}\|_{\dot{H}^{n/2}} + \|\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\nabla\boldsymbol{B}\|_{\dot{H}^{n/2}}).\end{aligned}$$

  Unfortunately it is impossible in this case to get rid of the second term on the right-hand side: in Section 5.2, we exhibit a counterexample to show that (1.4) does not hold in the case $s = n/2$, at least for $n = 2$, even if $\boldsymbol{u}$ and $\boldsymbol{B}$ are required to be divergence-free.

- In Chapter 6 we consider the MHD equations (1.1) with $\nu > 0$ and $\eta = 0$; i.e.,

  $$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.5a}$$
  $$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.5b}$$
  $$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.5c}$$

  on the whole of $\mathbb{R}^n$, with divergence-free initial data $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$ for $s > n/2$. We prove that there exists a time $T_* = T_*(s, \|\boldsymbol{u}_0\|_{H^s}, \|\boldsymbol{B}_0\|_{H^s}) > 0$ such that the equations (1.5) have a unique solution $\boldsymbol{u}, \boldsymbol{B} \in C([0, T_*]; H^s(\mathbb{R}^n))$.

  We also prove a similar result for the Stokes-MHD system (1.2) with $\nu > 0$

and $\eta = 0$; that is,

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.6a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.6b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.6c}$$

The main difficulty for both these models comes from the $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ term: the naive approach using the fact that H^s is an algebra would require control over $\|\nabla\boldsymbol{B}\|_{H^s}$, but there is no smoothing in the $\boldsymbol{B}$ equation. The key ingredient in proving the necessary a priori estimates is the inequality (1.4): given $\boldsymbol{u}, \boldsymbol{B}$ with $\nabla\boldsymbol{u}, \boldsymbol{B} \in H^s(\mathbb{R}^n)$ and $\nabla\cdot\boldsymbol{u} = 0$, (1.4) implies that

$$|\langle\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}], \Lambda^s\boldsymbol{B}\rangle| \le c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}^2,$$

alleviating the need to estimate $\|\nabla\boldsymbol{B}\|_{H^s}$.

- In Chapter 7, we consider again the MHD system (1.5) on all of $\mathbb{R}^n$, with divergence-free initial data $\boldsymbol{u}_0 \in B^{n/2-1}_{2,1}$ and $\boldsymbol{B}_0 \in B^{n/2}_{2,1}$. We prove that there exists a time $T_* = T_*(\nu, \boldsymbol{u}_0, \|\boldsymbol{B}_0\|_{B^{n/2}_{2,1}}) > 0$ such that the equations (1.5) have at least one weak solution $(\boldsymbol{u}, \boldsymbol{B})$, with

$$\begin{aligned}\boldsymbol{u} &\in L^\infty(0, T_*; B^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0, T_*; B^{n/2+1}_{2,1}(\mathbb{R}^n)),\\ \boldsymbol{B} &\in L^\infty(0, T_*; B^{n/2}_{2,1}(\mathbb{R}^n)).\end{aligned}$$

  The a priori estimates are valid in $\mathbb{R}^n$ for $n = 2, 3$; however, they include the term

$$\int_0^t \|\boldsymbol{u}(s)\|_{H^{n/2}}^2 \,\mathrm{d}s$$

  on the right-hand side, which thus requires an auxiliary bound in $H^{n/2-1}$. In 2D, this is simply achieved using the standard energy inequality, and the proof can be closed up quite easily. In 3D, however, we an auxiliary estimate in $H^{1/2}$ is required, which we prove using the splitting method of Calderón (1990). Finally, we prove that such solutions are unique in 3D; surprisingly, the proof of uniqueness in 2D is more difficult and remains open.

- Finally, in Chapter 8, we outline some conclusions and state some open problems of interest.

Chapter 2

A Tour Through the Zoo of Function Spaces

Throughout this thesis, we will need a wide variety of function spaces, including Lorentz, Sobolev and Besov spaces. Since a number of these spaces are not universally known, and some have more than one equivalent definition, this chapter consists of a brief review of all the spaces we will use. Almost all the material herein can be found in the books of Grafakos (2008, 2009) and Bahouri, Chemin & Danchin (2011).

2.1 Lebesgue and Lorentz Spaces $L^{p,q}$

Given a measurable subset $\Omega \subset \mathbb{R}^n$, the *Lebesgue space* $L^p(\Omega)$ is defined by

$$L^p := \left\{ f\colon \Omega \to \mathbb{R}^n : \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{1/p} < \infty \right\},$$

with

$$\|f\|_{L^p} := \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{1/p}$$

(with $\mathrm{d}x$ denoting integration with respect to the standard Lebesgue measure). There are two related generalisations of this which we consider now: the *weak L^p spaces*, denoted $L^{p,\infty}$, and the *Lorentz spaces*, denoted $L^{p,q}$.

2.1.1 Weak L^p Spaces

The weak L^p spaces are defined as follows (see, e.g., Grafakos (2008), §1.1). Let $\Omega \subset \mathbb{R}^n$ be measurable. Given a measurable, a.e.-finite function $f\colon \Omega \to \mathbb{R}$, we define its *distribution function* $d_f\colon [0,\infty) \to [0,\infty]$ by

$$d_f(\alpha) := \mu\{x \in \Omega : |f(x)| > \alpha\}.$$

Then, given $1 \le p < \infty$, the *weak L^p space*, denoted $L^{p,\infty}(\Omega)$, consists of all measurable, a.e.-finite functions f for which the quantity

$$\begin{aligned}\|f\|_{p,\infty} &= \inf\left\{C > 0 : d_f(\alpha) \le \frac{C^p}{\alpha^p} \text{ for all } \alpha > 0\right\} \\ &= \sup\{\gamma d_f(\gamma)^{1/p} : \gamma > 0\}\end{aligned}$$

is finite (see Grafakos (2008), Definition 1.1.5). It follows immediately from the definition that

$$f \in L^{p,\infty}(\Omega) \implies d_f(\alpha) \le \|f\|^p_{L^{p,\infty}} \alpha^{-p} \tag{2.1}$$

Note that $\|\cdot\|_{p,\infty}$ is *not* a norm, but only a quasi-norm — the triangle inequality fails to hold, but instead we have the replacement inequality

$$\|f + g\|_{L^{p,\infty}} \le 2(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}). \tag{2.2}$$

This follows from the fact that

$$d_{f+g}(\alpha) \le d_f(\alpha/2) + d_g(\alpha/2), \tag{2.3}$$

The following simple lemma is fundamental and shows that any function in L^p is also in $L^{p,\infty}$.

Lemma 2.1. *If $f \in L^p(\Omega)$ then $f \in L^{p,\infty}(\Omega)$ and $\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}$.*

Proof. The proof is essentially the proof of Chebyshev's inequality: simply note that

$$\|f\|^p_{L^p} = \int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \ge \int_{\{x : |f(x)| > \alpha\}} |f(x)|^p \, \mathrm{d}x \ge \alpha^p d_f(\alpha),$$

so if $f \in L^p(\Omega)$ then $d_f(\alpha) \le \|f\|^p_{L^p}\alpha^{-p}$, and hence $f \in L^{p,\infty}(\Omega)$. □

In fact we can use the distribution function to give an useful expression for the L^p norm of a function: it follows using Fubini's Theorem that

$$\|f\|^p_{L^p} = \int_\Omega |f(x)|^p \, \mathrm{d}x = p \int_\Omega \int_0^{|f(x)|} \alpha^{p-1} \, \mathrm{d}\alpha \, \mathrm{d}x = p \int_0^\infty \alpha^{p-1} d_f(\alpha) \, \mathrm{d}\alpha. \tag{2.4}$$

However, while Lemma 2.1 shows that $L^p(\Omega) \subset L^{p,\infty}(\Omega)$, it is in fact a proper subset, since the function $f(x) = |x|^{-n/p}$ is in $L^{p,\infty}(\mathbb{R}^n)$, even though it is clearly not in $L^p(\mathbb{R}^n)$: notice that

$$d_f(\alpha) = \mu\{x \in \mathbb{R}^n : |x|^{-n/p} > \alpha\} = \mu\{x \in \mathbb{R}^n : |x| < \alpha^{-p/n}\} = \omega_n \alpha^{-p},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, so $\|f\|_{L^{p,\infty}} = \omega_n^{1/p}$. So indeed $L^p(\mathbb{R}^n) \subsetneq L^{p,\infty}(\mathbb{R}^n)$.

An immediate indication of why these spaces are useful is given in the following simple result, which shows that in the L^p interpolation inequality

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}, \qquad \text{where } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

one can replace the Lebesgue spaces on the right-hand side by their weak counterparts.

Lemma 2.2. *Take $1 \leq p < r < q \leq \infty$. There exists a constant $c = c_{p,r,q}$ such that, for any $f \in L^{p,\infty} \cap L^{q,\infty}$, we have $f \in L^r$ and*

$$\|f\|_{L^r} \leq c_{p,r,q} \|f\|_{L^{p,\infty}}^{\theta} \|f\|_{L^{q,\infty}}^{1-\theta},$$

where $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. If $q = \infty$ we interpret $L^{\infty,\infty}$ as L^∞.

Proof. The proof can be found in Grafakos (2008), Proposition 1.1.14. □

2.1.2 Young's Inequality

One of the primary results for convolutions is Young's inequality.

Lemma 2.3 (Young's inequality)**.** *Let $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then for all $f \in L^q$, $g \in L^r$, we have $f \star g \in L^p$ with*

$$\|f \star g\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^r}. \tag{2.5}$$

We will need a version of this inequality that allows L^q on the right-hand side to be replaced by $L^{q,\infty}$. The price we have to pay for this is that we also weaken the left-hand side; and note that we also lose the possibility of some endpoint values ($r = \infty$ and $p, q = 1, \infty$) that are allowed in (2.5).

Proposition 2.4. *Let $1 \leq r < \infty$ and $1 < p, q < \infty$ satisfy $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then there exists a constant $c_{p,q,r} > 0$ such that, for all $f \in L^{q,\infty}$ and all $g \in L^r$, we have $f \star g \in L^{p,\infty}$ with*

$$\|f \star g\|_{L^{p,\infty}} \leq c_{p,q,r} \|f\|_{L^{q,\infty}} \|g\|_{L^r}. \tag{2.6}$$

Proof. The proof can be found in Grafakos (2008), Theorem 1.2.13. □

2.1.3 Lorentz Spaces

We now define the Lorentz spaces, which generalise both L^p and weak L^p spaces (see, e.g., Grafakos (2008), §1.4). Given a complex-valued measurable function f on (X, μ), we define $f^*\colon [0, \infty) \to [0, \infty]$, the *decreasing rearrangement* of f, by

$$f^*(t) := \inf\{s > 0 : d_f(s) \leq t\},$$

where $\inf \varnothing = \infty$. The point of this definition is that f and f^* have the same distribution function,

$$d_{f^*}(\alpha) = d_f(\alpha),$$

but f^* is a positive non-increasing scalar function. Since their distribution functions agree, we can use the identity in (2.4) to show that the L^p norm of f is equal to the L^p norm of f^* (see Grafakos (2008), Proposition 1.4.5, part (14)):

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x &= p \int_0^\infty \alpha^{p-1} d_f(\alpha) \,\mathrm{d}\alpha \\ &= p \int_0^\infty \alpha^{p-1} d_{f^*}(\alpha) \,\mathrm{d}\alpha = \int_0^\infty f^*(\alpha)^p \,\mathrm{d}\alpha. \end{aligned} \tag{2.7}$$

Given such an f and $1 \leq p, q \leq \infty$, define

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty. \end{cases}$$

The *Lorentz space* of all f with $\|f\|_{L^{p,q}} < \infty$ is denoted by $L^{p,q}(X, \mu)$.

It follows immediately from (2.7) that $L^{p,p}(X, \mu)$ is just the usual L^p space. It is also easy to show (see Grafakos (2008), Proposition 1.4.5, part (16)) that

$$\sup_{t>0} t^q f^*(t) = \sup_{\alpha>0} \alpha (d_f(\alpha))^q$$

for $0 < q < \infty$, and thus this definition of $L^{p,\infty}(X, \mu)$ agrees with the previous definition of a weak L^p space.

Since $L^p \subsetneq L^{p,\infty}$, it is natural to ask if the $L^{p,q}$ are similarly nested. This is indeed the case: $L^{p,r} \subset L^{p,s}$ whenever $r < s$; so the largest space in this family for fixed p is the weak space $L^{p,\infty}$, and the smallest is $L^{p,1}$. A proof may be found in Grafakos (2008), Proposition 1.4.10.

2.2 Sobolev Spaces $W^{k,p}$ and $\dot{H}^s$

2.2.1 Weak Derivatives

Let $\Omega \subset \mathbb{R}^n$ be measurable. For a smooth function ϕ, let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ be a multi-index, and define

$$\partial^\alpha \phi := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} \phi.$$

Suppose $u, v \in L^1_{\text{loc}}(\Omega)$ and α is a multi-index. We say that v is the α^{th} *weak partial derivative of* u, and write $\partial^\alpha u = v$, if

$$\int_\Omega u \partial^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \phi \, \mathrm{d}x$$

for all test functions $\phi \in C_c^\infty(\Omega)$. Here, as usual for multi-indices, $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

We may then define the Sobolev space

$$\begin{aligned} W^{k,p}(\Omega) := \{ f \in L^1_{\text{loc}}(\Omega) : &\text{ for every multi-index } \alpha \text{ with } |\alpha| \le k, \\ &\partial^\alpha f \text{ exists in the weak sense and belongs to } L^p(\Omega) \}. \end{aligned}$$

More details may be found in Evans (1998), Chapter 5.

We quote from Evans (1998) the general form of the Sobolev embedding theorem which we will use a number of times throughout the course of the thesis.

Theorem 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume that $u \in W^{k,p}(\Omega)$.*

(i) If $k < n/p$, then $u \in L^q(\Omega)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. More precisely, there is a constant C, depending only on k, p, n and Ω, such that

$$\|u\|_{L^q} \le C \|u\|_{W^{k,p}}.$$

(ii) If $k > n/p$, then $u \in L^\infty(\Omega)$, and there is a constant C, depending only on k, p, n and Ω, such that

$$\|u\|_{L^\infty} \le C \|u\|_{W^{k,p}}.$$

For a proof, see Theorem 6 in Section 5.6 of Evans (1998). Note that Theorem 2.5 does not cover the case $k = n/p$; we will return to this in Section 2.3.

2.2.2 The Fourier Transform

The *Schwartz space* $\mathscr{S}$ consists of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi| \le M_{\alpha,\beta} \qquad \text{for all } \alpha, \beta \ge 0,$$

where α, β are multi-indices. For any $f \in \mathscr{S}$ one can define the Fourier transform

$$\mathscr{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x} f(x)\,\mathrm{d}x. \tag{2.8}$$

It is straightforward to check that

$$\mathscr{F}[\partial^\alpha f](\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \hat{f}(\xi) \qquad \text{and} \qquad \mathscr{F}[x^\beta f](\xi) = (-2\pi i)^{|\beta|}[\partial^\beta \hat{f}](\xi),$$

from which it follows that $\mathscr{F}$ maps $\mathscr{S}$ into itself.

Given the Fourier transform of f, one can reconstruct f as follows:

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi\cdot x} \hat{f}(\xi)\,\mathrm{d}\xi. \tag{2.9}$$

By defining $\sigma(f)$ by $\sigma(f)(x) = f(-x)$ we can write the inversion formula more compactly as $f = \sigma \circ \mathscr{F}(\hat{f})$. We define $\mathscr{F}^{-1} = \sigma \circ \mathscr{F}$, in order that when we can meaningfully extend the definition of $\mathscr{F}$ and σ we will retain this inversion formula.

An obvious extension of the Fourier transform is to any function $f \in L^1(\mathbb{R}^n)$, using the integral definition in (2.8) directly. Since

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}x = \|f\|_{L^1}$$

it follows that $\mathscr{F}$ maps L^1 into L^∞. Furthermore, there is a natural definition of the Fourier transform for $f \in L^2(\mathbb{R}^n)$. Given $f \in \mathscr{S}$,

$$\begin{aligned} \|\hat{f}\|_{L^2}^2 &= \int_{\mathbb{R}^n} \overline{\hat{f}(x)} \left(\int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x} f(\xi)\,\mathrm{d}\xi \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(\xi) \left(\int_{\mathbb{R}^n} \overline{\hat{f}(x) e^{2\pi i \xi\cdot x}}\,\mathrm{d}x \right) \mathrm{d}\xi \\ &= \int \overline{f(\xi)} f(\xi)\,\mathrm{d}\xi = \|f\|_{L^2}^2. \end{aligned} \tag{2.10}$$

Now given any $f \in L^2$, one can write $f = \lim_{n\to\infty} f_n$, where $f_n \in \mathscr{S}$ and the limit is taken in L^2. It follows that $\hat{f}_n$ is Cauchy in L^2, and we identify its limit as $\hat{f}$. So we can define $\mathscr{F}\colon L^2 \to L^2$, with $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

The Fourier transform can therefore be defined (by linearity) for any $f \in L^1 + L^2$; f can be recovered from $\hat{f}$ using $\mathscr{F}^{-1}$ if $\hat{f} \in L^1 + L^2$, and if $\hat{f} \in L^1$ (in particular if $\hat{f} \in \mathscr{S}$) then we can use the Fourier inversion formula (2.9) to give f pointwise as an integral involving $\hat{f}$. By writing $f \in L^{r,\infty}$ as $f = f_{M-} + f_{M+}$, where

$$f_{M-} = f\mathbb{1}_{\{|f| \le M\}} \quad \text{and} \quad f_{M+} = f\mathbb{1}_{\{|f| > M\}},$$

it is easy to show that $f_{M-} \in L^p$ and $f_{M+} \in L^q$ for any $1 \le q < r < p \le \infty$ (see the proof of Theorem 1.2.13 in Grafakos (2008)). Hence one may define the Fourier transform if $f \in L^{r,\infty}$ for some $1 < r < 2$ (and in particular if $f \in L^r$), by splitting f into two parts, one in L^1 and one in L^2.

One can extend the definition further to the space of tempered distributions $\mathscr{S}'$. We say that a sequence $\{\phi_n\} \in \mathscr{S}$ converges to $\phi \in \mathscr{S}$ if

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta (\phi_n - \phi)| \to 0 \qquad \text{for all } \alpha, \beta \ge 0,$$

and a linear functional F on $\mathscr{S}$ is an element of $\mathscr{S}'$ if $\langle F, \phi_n \rangle \to \langle F, \phi \rangle$ whenever $\phi_n \to \phi$ in $\mathscr{S}$. It is easy to show that, for any $\phi, \psi \in \mathscr{S}$,

$$\langle \phi, \hat{\psi} \rangle = \langle \hat{\phi}, \psi \rangle,$$

and this[1] allows us to define the Fourier transform for $F \in \mathscr{S}'$ by setting

$$\langle \hat{F}, \psi \rangle = \langle F, \hat{\psi} \rangle \qquad \text{for every } \psi \in \mathscr{S}.$$

Since one can also extend the definition of σ to $\mathscr{S}'$ via the definition $\langle \sigma(F), \psi \rangle = \langle F, \sigma(\psi) \rangle$, the identity $F = \mathscr{F}^{-1}\hat{F}$ still holds in this generality.

2.2.3 Homogeneous and Inhomogeneous Sobolev Spaces

Since the Fourier transform maps L^2 isometrically into itself (by Plancherel's formula (2.10)), it is relatively straightforward to show that when s is a non-negative integer

$$\sum_{|\alpha| = s} \|\partial^\alpha f\|_{L^2}^2 \simeq \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi, \tag{2.11}$$

$$\sum_{|\alpha| \le s} \|\partial^\alpha f\|_{L^2}^2 \simeq \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi, \tag{2.12}$$

[1]We use $\langle \cdot, \cdot \rangle$ for the action of an element of $\mathscr{S}'$ on elements of $\mathscr{S}$, and set $\langle f, g \rangle = \int fg$ when f and g are functions.

where we write $a \simeq b$ if there are constants $0 < c \leq C$ such that $ca \leq b \leq Ca$.

For any $s \geq 0$, even if s is not an integer, we can define[2] the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ using (2.11):

$$\dot{H}^s(\mathbb{R}^n) = \left\{ f \in \mathscr{S}' : \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi < \infty \right\}.$$

For $s < n/2$ this is a Hilbert space with the natural norm

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2},$$

and one can therefore also define $\dot{H}^s(\mathbb{R}^n)$ in this case as the completion of $\mathscr{S}$ with respect to the $\dot{H}^s$ norm (that $\dot{H}^s(\mathbb{R}^n)$ is complete iff $s < n/2$ is shown in Bahouri et al. (2011); the simple example showing that $\dot{H}^s(\mathbb{R}^n)$ is not complete when $s \geq n/2$ can also be found in Chemin et al. (2006)).

Similarly, using (2.12) we can define the inhomogeneous Sobolev space $H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathscr{S}' : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi < \infty \right\}.$$

For any $s \in \mathbb{R}$ this is a Hilbert space with the natural norm

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}.$$

By (2.12) we see that $W^{s,2} \cong H^s$ (i.e. their norms are equivalent).

We can also define fractional derivative operators J^s and Λ^s in terms of Fourier transforms as follows:

$$\mathscr{F}[J^s f](\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi), \qquad \mathscr{F}[\Lambda^s f](\xi) = |\xi|^s \hat{f}(\xi). \tag{2.13}$$

Plancherel's formula (2.10) guarantees that

$$\|J^s f\|_{L^2} = \|f\|_{H^s}, \qquad \|\Lambda^s f\|_{L^2} = \|f\|_{\dot{H}^s}.$$

[2] We follow the definition of Bahouri et al. (2011) (see also Chemin, Desjardins, Gallagher & Grenier (2006)), including the condition that $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. This sidesteps complexities that arise from problems with understanding the meaning of $|\xi|^s \hat{f}$ if one only knows that $\hat{f} \in \mathscr{S}'$; see the discussion in Chapter 6 of Grafakos (2009).

2.2.4 The Endpoint Sobolev Embedding

We now give a simple proof of the Sobolev embedding theorem for homogeneous Sobolev spaces: namely that in part (i) of Theorem 2.5 we may replace the $W^{k,p}$ norm with the $\dot{H}^s$ norm for the correct value of s. While the method of proof is not new, we will use the same method of splitting the function in Chapter 3.

Theorem 2.6. *For $2 < p < \infty$ there exists a constant $c = c_{n,p}$ such that if $f \in \dot{H}^s(\mathbb{R}^n)$ with $s = n(1/2 - 1/p)$ then $f \in L^p(\mathbb{R}^n)$ and*

$$\|f\|_{L^p} \leq c\|f\|_{\dot{H}^s}. \tag{2.14}$$

Proof. We follow the proof in Chemin et al. (2006), Theorem 1.2. First we prove the result when $\|f\|_{\dot{H}^s} = 1$. For such an f, write $f = f_{<R} + f_{>R}$, where

$$f_{<R} = \mathscr{F}^{-1}(\hat{f}\chi_{\{|\xi|\leq R\}}) \qquad \text{and} \qquad f_{>R} = \mathscr{F}^{-1}(\hat{f}\chi_{\{|\xi|>R\}}). \tag{2.15}$$

In both expressions the Fourier inversion formula makes sense: for $f_{>R}$ we know that $\hat{f}\chi_{>R} \in L^2(\mathbb{R}^n)$, and $\mathscr{F}$ (and likewise $\mathscr{F}^{-1}$) is defined on L^2; while for $f_{<R}$ we know that $\hat{f} \in L^1_{\text{loc}}(\mathbb{R}^n)$, and so $\hat{f}\chi_{\leq R} \in L^1(\mathbb{R}^n)$ which means that we can write $f_{<R}$ using the integral form of the inversion formula (2.9) to write

$$f_{<R}(x) = \int_{|\xi|\leq R} e^{2\pi i\xi\cdot x}\hat{f}(\xi)\,\mathrm{d}\xi.$$

Thus

$$\begin{aligned}\|f_{<R}\|_{L^\infty} &\leq \int_{|\xi|\leq R} |\xi|^{-s}|\xi|^s|\hat{f}(\xi)|\,\mathrm{d}\xi \\ &\leq \left(\int_{|\xi|\leq R} |\xi|^{-2s}\,\mathrm{d}\xi\right)^{1/2} \|f\|_{\dot{H}^s} = C_s R^{n/2-s} = C_s R^{n/p},\end{aligned}$$

since we took $\|f\|_{\dot{H}^s} = 1$ and $s = n(\frac{1}{2} - \frac{1}{p})$. Now, since for any choice of R

$$d_f(\alpha) \leq d_{f_{<R}}(\alpha/2) + d_{f_{>R}}(\alpha/2)$$

(using (2.3)), we can choose R to depend on α, $R = R_\alpha := (\alpha/2C_s)^{p/n}$, and then we have

$$d_{f_{<R_\alpha}}(\alpha/2) = 0.$$

Hence it follows that $d_f(\alpha) \leq d_{f_{>R_\alpha}}(\alpha/2)$. Thus, using the fact that the Fourier transform is an isometry from L^2 into itself,

$$\begin{aligned}
\|f\|_{L^p}^p &\le p\int_0^\infty \alpha^{p-1} d_{f_{>R_\alpha}}(\alpha/2)\,\mathrm{d}\alpha \\
&\le p\int_0^\infty \alpha^{p-1}\frac{4}{\alpha^2}\|f_{>R_\alpha}\|_{L^2}^2\,\mathrm{d}\alpha \\
&= C\int_0^\infty \alpha^{p-3}\|\mathscr{F}(f_{>R_\alpha})\|_{L^2}^2\,\mathrm{d}\alpha \\
&= C\int_0^\infty \alpha^{p-3}\int_{|\xi|\ge R_\alpha}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}\alpha \\
&= C\int_{\mathbb{R}^n}\left(\int_0^{2C_s|\xi|^{n/p}}\alpha^{p-3}\,\mathrm{d}\alpha\right)|\hat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\le C\int_{\mathbb{R}^n}|\xi|^{n(p-2)/p}|\hat{f}(\xi)|^2\,\mathrm{d}s \\
&= C,
\end{aligned}$$

since $n(p-2)/p = 2s$ and we took $\|f\|_{\dot{H}^s} = 1$.

Thus for $f \in \dot{H}^s$ with $\|f\|_{\dot{H}^s} = 1$ we have $\|f\|_{L^p} \le C$, and (2.14) follows for general $f \in \dot{H}^s$ on applying this result to $g = f/\|f\|_{\dot{H}^s}$. □

2.3 Bounded Mean Oscillation BMO

For any set $A \subset \mathbb{R}^n$ we write

$$f_A = \frac{1}{|A|}\int_A f\,\mathrm{d}x$$

for the average of f over the set A. The space of functions with *bounded mean oscillation*, BMO($\mathbb{R}^n$), consists of those functions f for which

$$\|f\|_{\mathrm{BMO}} := \sup_{Q\subset\mathbb{R}^n}\frac{1}{|Q|}\int_Q |f - f_Q|\,\mathrm{d}x$$

is finite, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Note that this is a not a norm (any constant function has $\|c\|_{\mathrm{BMO}} = 0$), but BMO is a linear space, i.e. if $f, g \in$ BMO then $f + g \in$ BMO and

$$\|f + g\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}} + \|g\|_{\mathrm{BMO}}.$$

This space was introduced by John & Nirenberg (1961); more details can be found in Chapter 7 of Grafakos (2009), for example.

BMO is a space with the same scaling as L^∞, but is a larger space. Indeed, if $f \in L^\infty(\mathbb{R}^n)$ then clearly for any cube Q

$$\int_Q |f - f_Q| \, \mathrm{d}x \leq 2 \int_Q |f| \leq 2|Q| \|f\|_{L^\infty}, \tag{2.16}$$

and so

$$\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}. \tag{2.17}$$

However, the function $\log|x| \in \mathrm{BMO}(\mathbb{R}^n)$ but is not bounded on $\mathbb{R}^n$ (Example 7.1.3 in Grafakos (2009)).

We now prove a simple lemma showing that we can replace the cubes in the definition of BMO with balls.

Lemma 2.7. *There exists a constant $C = C(n)$ such that*

$$\|f\|_{\mathrm{BMO}} \leq C \sup_B \frac{1}{|B|} \int_B |f - f_B| \, \mathrm{d}x, \tag{2.18}$$

where the supremum is taken over all balls B.

Proof. The proof is based on Grafakos (2009), Proposition 7.1.2, part (8). Any cube Q is contained in a ball B such that $|B|/|Q| = 2^{-n}\omega_n n^{n/2}$, and so

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f(x) - f_B| \, \mathrm{d}x &\leq \frac{|B|}{|Q|} \frac{1}{|B|} \int_B |f(x) - f_B| \, \mathrm{d}x \\ &\leq 2^{-n}\omega_n n^{n/2} \sup_B \frac{1}{|B|} \int_B |f - f_B| \, \mathrm{d}x. \end{aligned} \tag{2.19}$$

This is not quite (2.18); but note that for any cube Q

$$|f - f_Q| \leq |f - f_B| + |f_B - f_Q| \leq |f - f_B| + \frac{1}{|Q|} \int_Q |f - f_B|,$$

and so

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq 2 \frac{1}{|Q|} \int_Q |f - f_B|,$$

which coupled with (2.19) yields the required result. □

By using a particular form of Poincaré's inequality, namely

$$\|u - u_{B(x,r)}\|_{L^1(B(x,r))} \leq Cr \|Du\|_{L^1(B(x,r))}$$

(for a proof see Evans (1998), Section 5.8, Theorem 2, for example), we can show the embedding $W^{1,n} \subset \mathrm{BMO}$ (for $n \geq 2$):

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)} |u - u_{B(x,r)}| &\leq Cr\frac{1}{|B(x,r)|}\int_{B(x,r)} |Du|\,\mathrm{d}y \\ &\leq Cr\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |Du|^n\,\mathrm{d}y\right)^{1/n} \\ &\leq C\left(\int_{\mathbb{R}^n} |Du|^n\,\mathrm{d}y\right)^{1/n},\end{aligned}$$

thus using Lemma 2.7 we obtain $\|u\|_{\mathrm{BMO}} \leq C\|u\|_{W^{1,n}}$.

Similarly, the endpoint Sobolev embedding from Theorem 2.6 fails when $s = n/2$, but at this endpoint we still have $\dot{H}^{n/2}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$. This is simple to show (following Theorem 1.48 in Bahouri et al. (2011)), if we note that for any $x \in Q$

$$|f(x) - f_Q| = \left|\frac{1}{|Q|}\int_Q f(x) - f(y)\,\mathrm{d}y\right| \leq \sqrt{n}|Q|^{1/n}\|\nabla f\|_{L^\infty(Q)}.$$

Lemma 2.8. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \dot{H}^{n/2}(\mathbb{R}^n)$ then $f \in \mathrm{BMO}(\mathbb{R}^n)$ and there exists a constant $C = C(n)$ such that*

$$\|f\|_{\mathrm{BMO}} \leq C\|f\|_{\dot{H}^{n/2}} \qquad \textit{for all } f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \dot{H}^{n/2}(\mathbb{R}^n).$$

Proof. We write $f = f_{<R} + f_{>R}$ as in the proof of Theorem 2.6 and then, recalling (2.16),

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f - f_Q| &\leq \sqrt{n}|Q|^{1/n}\|\nabla f_{<R}\|_{L^\infty(Q)} + \frac{1}{|Q|}\int_Q |f_{>R} - (f_{>R})_Q| \\ &\leq \sqrt{n}|Q|^{1/n}\int_{|\xi|\leq R} |\xi||\hat{f}(\xi)|\,\mathrm{d}\xi + \frac{2}{|Q|^{1/2}}\left(\int_Q |f_{<R}|^2\right)^{1/2} \\ &\leq \sqrt{n}|Q|^{1/n}cR\left(\int_{\mathbb{R}^n} |\xi|^{n/2}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1/2} + \frac{2}{|Q|^{1/2}}\left(\int_{|\xi|\geq R} |\hat{f}(\xi)|^2\right)^{1/2} \\ &\leq c_n[|Q|^{1/n}R + |Q|^{-1/2}R^{n/2}]\|f\|_{\dot{H}^{n/2}}.\end{aligned}$$

Choosing $R = |Q|^{-1/n}$ yields

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq C\|f\|_{\dot{H}^{n/2}};$$

taking the supremum over all cubes $Q \subset \mathbb{R}^n$ yields $\|f\|_{\mathrm{BMO}} \leq C\|f\|_{\dot{H}^{n/2}}$. $\square$

2.4 Besov Spaces $B^s_{p,r}$ and $\dot{B}^s_{p,r}$

Here we recall some of standard theory of Besov spaces which we will use in Chapter 7; we mostly use the same notation as Bahouri et al. (2011), and refer the reader to Chapter 2 therein for proofs and many more details that we must omit.

2.4.1 Definitions

For the purposes of this section, given a function ϕ and $j \in \mathbb{Z}$ we denote by ϕ_j the dilation

$$\phi_j(\xi) = \phi(2^{-j}\xi).$$

Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$. There exist radial functions[3] $\chi \in C^\infty_c(B(0,4/3))$ and $\varphi \in C^\infty_c(\mathcal{C})$ both taking values in $[0,1]$ such that

$$\text{for all } \xi \in \mathbb{R}^n, \qquad \chi(\xi) + \sum_{j\geq 0} \varphi_j(\xi) = 1, \tag{2.20a}$$

$$\text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \qquad \sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \tag{2.20b}$$

$$\text{if } |j - j'| \geq 2, \text{ then} \qquad \operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_{j'} = \varnothing, \tag{2.20c}$$

$$\text{if } j \geq 1, \text{ then} \qquad \operatorname{supp}\chi \cap \operatorname{supp}\varphi_j = \varnothing; \tag{2.20d}$$

the set $\widetilde{\mathcal{C}} := B(0,2/3) + \mathcal{C}$ is an annulus, and

$$\text{if } |j - j'| \geq 5, \text{ then} \qquad 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \varnothing. \tag{2.20e}$$

Furthermore, we have

$$\text{for all } \xi \in \mathbb{R}^n, \qquad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi_j^2(\xi) \leq 1, \tag{2.20f}$$

$$\text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \qquad \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi_j^2(\xi) \leq 1. \tag{2.20g}$$

As in Section 2.2.2 earlier, denote by $\mathscr{F}[u]$ the Fourier transform of u, and let $h = \mathscr{F}^{-1}\varphi$ and $\widetilde{h} = \mathscr{F}^{-1}\chi$. Given a measurable function σ defined on $\mathbb{R}^n$ with at most polynomial growth at infinity, we define the Fourier multiplier operator M_σ by $M_\sigma u := \mathcal{F}^{-1}(\sigma\hat{u})$.

[3] Note that χ is *not* a characteristic function.

For $j \in \mathbb{Z}$, the *inhomogeneous dyadic blocks* $\triangle_j$ are defined as follows:

$$\begin{aligned}
&\text{if } j \leq -2, && \triangle_j u = 0,\\
& && \triangle_{-1} u = M_\chi u = \int_{\mathbb{R}^n} \widetilde{h}(y) u(x-y)\,\mathrm{d}y,\\
&\text{if } j \geq 0, && \triangle_j u = M_{\varphi_j} u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y)\,\mathrm{d}y.
\end{aligned}$$

The inhomogeneous low-frequency cut-off operator S_j is defined by

$$S_j u := \sum_{j' \leq j-1} \triangle_{j'} u.$$

For $j \in \mathbb{Z}$, the *homogeneous dyadic blocks* $\dot{\triangle}_j$ and the homogeneous low-frequency cut-off operator $\dot{S}_j$ are defined as follows:

$$\begin{aligned}
\dot{\triangle}_j u &= M_{\varphi_j} u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y)\,\mathrm{d}y,\\
\dot{S}_j u &= M_{\chi_j} u = 2^{jn} \int_{\mathbb{R}^n} \widetilde{h}(2^j y) u(x-y)\,\mathrm{d}y.
\end{aligned}$$

Formally, we can write the following *Littlewood–Paley decompositions*:

$$\mathrm{Id} = \sum_{j \in \mathbb{Z}} \triangle_j \qquad \text{and} \qquad \mathrm{Id} = \sum_{j \in \mathbb{Z}} \dot{\triangle}_j.$$

In the inhomogeneous case, the decomposition makes sense in $\mathscr{S}'(\mathbb{R}^n)$: if $u \in \mathscr{S}'(\mathbb{R}^n)$ is a tempered distribution, then $u = \lim_{j \to \infty} S_j u$ in $\mathscr{S}'(\mathbb{R}^n)$. Unfortunately, the homogeneous case is a little more involved. We denote by $\mathscr{S}'_h(\mathbb{R}^n)$ the space of tempered distributions such that

$$\lim_{\lambda \to \infty} \|M_{\theta(\lambda\,\cdot)} u\|_{L^\infty} = 0 \quad \text{ for any } \theta \in C_c^\infty(\mathbb{R}^n).$$

Then the homogeneous decomposition makes sense in $\mathscr{S}'_h(\mathbb{R}^n)$: if $u \in \mathscr{S}'_h(\mathbb{R}^n)$, then $u = \lim_{j \to \infty} \dot{S}_j u$ in $\mathscr{S}'_h(\mathbb{R}^n)$. Moreover, using the homogeneous decomposition, it is straightforward to show that

$$\dot{S}_j u = \sum_{j' \leq j-1} \dot{\triangle}_{j'} u.$$

Given a real number s and two numbers $p, r \in [1, \infty]$, the *homogeneous Besov* *space* $\dot{B}^s_{p,r}(\mathbb{R}^n)$ consists of those distributions u in $\mathscr{S}'_h(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\triangle}_j u\|^r_{L^p}\right)^{1/r} < \infty$$

if $r < \infty$, and

$$\|u\|_{\dot{B}^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\triangle}_j u\|_{L^p} < \infty$$

if $r = \infty$. This is a normed space, and its norm is independent of the choice of function φ used to define the blocks $\dot{\triangle}_j$. Note that a distribution $u \in \mathscr{S}'_h(\mathbb{R}^n)$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if, and only if, there exists a constant C and a non-negative sequence $(d_j)_{j\in\mathbb{Z}}$ such that

$$\text{for all } j \in \mathbb{Z}, \quad \|\dot{\triangle}_j u\|_{L^p} \le C d_j 2^{-js} \qquad \text{and} \qquad \|(d_j)\|_{\ell^r} = 1. \tag{2.21}$$

It follows immediately from (2.20g) that the seminorms $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{B}^s_{2,2}}$ are equivalent, and hence that $\dot{H}^s \subset \dot{B}^s_{2,2}$ and that both spaces coincide for $s < n/2$.

We also define the *inhomogeneous Besov space* $B^s_{p,r}(\mathbb{R}^n)$ as the space of those distributions u in $\mathscr{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{B^s_{p,r}} := \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\triangle_j u\|^r_{L^p}\right)^{1/r} < \infty$$

if $r < \infty$, and

$$\|u\|_{B^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{js}\|\triangle_j u\|_{L^p} < \infty$$

if $r = \infty$. It is straightforward to show that $B^s_{p,r} = \dot{B}^s_{p,r} \cap L^p$, and that $B^s_{p,r}$ is always a Banach space. For that reason, we focus mainly on homogeneous Besov spaces; most of the following results have inhomogeneous versions, which can be found in Sections 2.7 and 2.8 of Bahouri et al. (2011).

2.4.2 Embeddings

Much like the Sobolev embeddings, Besov spaces embed in certain L^p spaces with the correct exponents. We quote the two embeddings we will use most frequently.

Proposition 2.9 (Proposition 2.20 in Bahouri et al. (2011)). *Let* $1 \le p_1 \le p_2 \le \infty$ *and* $1 \le r_1 \le r_2 \le \infty$. *For any real number* s, *we have the continuous embedding*

$$\dot{B}^s_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,r_2}(\mathbb{R}^n).$$

Proposition 2.10 (Proposition 2.39 in Bahouri et al. (2011))**.** *For* $1 \leq p \leq q \leq \infty$*, we have the continuous embedding*

$$\dot{B}^{n/p-n/q}_{p,1}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

Note that the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ is a Banach space if, and only if, either $s < n/p$, or $s = n/p$ and $r = 1$ (in contrast to its inhomogeneous counterpart). Indeed, it is the case $\dot{B}^{n/p}_{p,1}$ that most interests us, especially when $p = 2$, for three reasons: it is a Banach space, it embeds continuously in $L^\infty(\mathbb{R}^n)$ by Proposition 2.10, and it is a Banach algebra. The last fact follows from Bony's paraproduct decomposition, which we outline now.

2.4.3 Homogeneous Paradifferential Calculus

Let u and v be tempered distributions in $\mathscr{S}'_h(\mathbb{R}^n)$. We have

$$u = \sum_{j' \in \mathbb{Z}} \dot{\triangle}_{j'} u \quad \text{and} \quad v = \sum_{j \in \mathbb{Z}} \dot{\triangle}_j v,$$

so, at least formally,

$$uv = \sum_{j,j' \in \mathbb{Z}} \dot{\triangle}_{j'} u \dot{\triangle}_j v.$$

Paradifferential calculus breaks the above sum into three parts, as follows: define

$$\dot{T}_u v := \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\triangle}_j v,$$

and

$$\dot{R}(u,v) := \sum_{|k-j| \leq 1} \dot{\triangle}_k u \dot{\triangle}_j v.$$

At least formally, the following *Bony decomposition* holds true:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v).$$

We now state two standard estimates on $\dot{T}$ and $\dot{R}$ that we will use in proving our a priori estimates in Section 7.1.

Lemma 2.11 (Theorem 2.47 from Bahouri et al. (2011))**.** *Let* $s \in \mathbb{R}$ *and* $t < 0$*. There exists a constant* $C = C(s,t)$ *such that for any* $p, r_1, r_2 \in [1, \infty]$*,* $u \in \dot{B}^t_{p,r_1}$ *and* $v \in \dot{B}^s_{p,r_2}$*,*

$$\|\dot{T}_u v\|_{\dot{B}^{s+t}_{p,r}} \leq C \|u\|_{\dot{B}^t_{\infty,r_1}} \|v\|_{\dot{B}^s_{p,r_2}}$$

with $\frac{1}{r} = \min\left\{1, \frac{1}{r_1} + \frac{1}{r_2}\right\}$.

Lemma 2.12 (Theorem 2.52 from Bahouri et al. (2011))**.** *Let* $s_1, s_2 \in \mathbb{R}$ *such that* $s_1 + s_2 > 0$. *There exists a constant* $C = C(s_1, s_2)$ *such that, for any* $p_1, p_2, r_1, r_2 \in [1, \infty]$, $u \in \dot{B}^{s_1}_{p_1,r_1}$ *and* $v \in \dot{B}^{s_2}_{p_2,r_2}$,

$$\|\dot{R}(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leq C\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}}$$

provided that

$$\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \textit{and} \quad \frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$

From Lemmas 2.11 and 2.12 it is straightforward to prove that, if $s > 0$ and $p, r \in [1, \infty]$ such that either $s < n/p$, or $s = n/p$ and $r = 1$, then there is a constant C depending only on s and the dimension n such that

$$\|uv\|_{\dot{B}^s_{p,r}} \leq C\left(\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|u\|_{\dot{B}^s_{p,r}}\|v\|_{L^\infty}\right).$$

In particular, $L^\infty \cap \dot{B}^s_{p,r}$ is a Banach algebra. Moreover, as $\dot{B}^{n/p}_{p,1}$ embeds continuously in L^∞ (by Proposition 2.10), we see that $\dot{B}^{n/p}_{p,1}$ is an algebra and

$$\|uv\|_{\dot{B}^{n/p}_{p,1}} \leq c\|u\|_{\dot{B}^{n/p}_{p,1}}\|v\|_{\dot{B}^{n/p}_{p,1}}. \tag{2.22}$$

Chapter 3

Sobolev Interpolation and Ladyzhenskaya's Inequality

In order to prove existence and uniqueness for our system (1.2), we will require a variant of Ladyzhenskaya's inequality. We first recall the standard inequality proved by Ladyzhenskaya (1958).

Lemma 3.1. *If $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, then for $u \in H^1(\Omega)$,*

$$\|u\|_{L^4} \le c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}. \tag{3.1}$$

Ladyzhenskaya first proved this inequality when looking at weak solutions of the Navier–Stokes equations:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned}$$

After multiplying by a test function ϕ and integrating by parts, one can use (3.1) to estimate the nonlinear term as follows:

$$|\langle(\boldsymbol{u}\cdot\nabla)\phi, \boldsymbol{u}\rangle| \le \|\boldsymbol{u}\|_{L^4}\|\nabla\phi\|_{L^2}\|\boldsymbol{u}\|_{L^4} \le \|\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\phi\|_{L^2}.$$

One can prove Ladyzhenskaya's inequality simply by using the Sobolev embedding $H^{1/2} \subset L^4$ and interpolating $H^{1/2}$ between L^2 and H^1:

$$\|u\|_{L^4} \le c\|u\|_{H^{1/2}} \le c\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}.$$

The following direct proof, which is essentially that of Ladyzhenskaya (1958), can be found in modern notation in Foias, Manley, Rosa & Temam (2001), equation (4.8) on p. 17, or Robinson (2001), Lemma 5.27.

Proof of Lemma 3.1. It suffices to prove the result for $u \in C_c^1(\Omega')$, where $\overline{\Omega} \Subset \Omega'$, since the result then follows by density of $C_c^1(\Omega')$ in $H^1(\Omega')$ and the extension theorem for Sobolev spaces. Because

$$[u(x)]^2 = 2\int_{-\infty}^{x_j} u\partial_j u \,\mathrm{d}y_j$$

for $j = 1, 2$, we have

$$\max_{x_j} |u(x)|^2 \le 2\int_{-\infty}^{\infty} |u\partial_j u| \,\mathrm{d}x_j.$$

Therefore

$$\begin{aligned}\iint_{\mathbb{R}^2} |u|^4 \,\mathrm{d}x_1 \,\mathrm{d}x_2 &\le \left(\int_{-\infty}^{\infty} \max_{x_2} |u(x)|^2 \,\mathrm{d}x_1\right)\left(\int_{-\infty}^{\infty} \max_{x_1} |u(x)|^2 \,\mathrm{d}x_2\right) \\ &\le 4\left(\iint_{\mathbb{R}^2} \max_{x_2} |u\partial_2 u| \,\mathrm{d}x_1\mathrm{d}x_2\right)\left(\iint_{\mathbb{R}^2} \max_{x_1} |u\partial_1 u| \,\mathrm{d}x_1\mathrm{d}x_2\right).\end{aligned}$$

Since $\iint_{\mathbb{R}^2} |u\partial_j u| \,\mathrm{d}x \le \|u\|_{L^2}\|\partial_j u\|_{L^2}$, we obtain

$$\|u\|_{L^4} \le 4^{1/4}\|u\|_{L^2}^{1/2}\|\partial_1 u\|_{L^2}^{1/4}\|\partial_2 u\|_{L^2}^{1/4} \le c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}. \qquad \square$$

In order to prove existence of weak solutions for the system

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.2a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.2b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.2c}$$

one can first perform a formal energy estimate to show that $\boldsymbol{B} \in L^\infty(0, T; L^2(\Omega))$ provided that $\boldsymbol{B}_0 \in L^2(\Omega)$. To obtain a similar estimate on $\boldsymbol{u}$, we must consider the regularity of the Stokes equations

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{3.2a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.2b}$$

when $\boldsymbol{B} \in L^2$ is given.

To simplify the problem, let us simply consider the Laplace equation

$$-\Delta\phi = \partial_k f \tag{3.3}$$

for $f \in L^1$, which has the same features. The solution of the Laplace equation (3.3) on the whole space $\mathbb{R}^2$ is given by convolution of the forcing (in this case $\partial_k f$) with the fundamental solution

$$E(x) = -\frac{1}{2\pi} \log |x|.$$

In other words, after an integration by parts, $\phi = \partial_k E \star f$. Since $\partial_k E \in L^{2,\infty}(\mathbb{R}^2)$ the weak version of Young's inequality (Proposition 2.4) implies that $\phi \in L^{2,\infty}(\mathbb{R}^2)$.

One thus expects that the solution of the Stokes equation (3.2) also belongs to $L^{2,\infty}$ whenever $\boldsymbol{B} \in L^2$. However, for that to be of use, we would require a version of Ladyzhenskaya's inequality with $\|u\|_{L^2}$ replaced with $\|u\|_{L^{2,\infty}}$: that is,

$$\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2} \|\nabla f\|_{L^2}^{1/2}. \tag{3.4}$$

In this chapter, we will offer two completely different proofs of inequality (3.4):

- a simple, concrete proof involving Fourier transforms;
- an elegant, quick proof using the abstract theory of interpolation spaces.

Using Fourier transforms we will prove the following generalisation of (3.4):

$$\|f\|_{L^p} \leq c\|f\|_{L^{q,\infty}}^{\theta} \|f\|_{\dot{H}^s}^{1-\theta} \tag{3.5}$$

for every $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$, as long as $1 < q < p$, $s \geq 0$ and $s > n\left(\frac{1}{2} - \frac{1}{p}\right)$, and

$$\frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right).$$

In the case $s = n/2$, we obtain a stronger version of (3.5) using the theory of interpolation spaces. In particular, we can replace the L^p norm on the left by the Lorentz space $L^{p,r}(\mathbb{R}^n)$, and the $\dot{H}^{n/2}$ norm on the right by the BMO norm, and obtain

$$\|f\|_{L^{p,r}} \leq c\|f\|_{L^{q,\infty}}^{q/p} \|f\|_{\mathrm{BMO}}^{1-q/p} \tag{3.6}$$

for every $f \in L^{q,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, when $1 < q < p < \infty$, and $1 \leq r \leq \infty$.

The inequality (3.6) is not altogether new: an alternative proof is sketched in Kozono, Minamidate & Wadade (2007). Furthermore, it is a strengthening of the inequality

$$\|f\|_{L^p} \leq c\|f\|_{L^q}^{q/p} \|f\|_{\mathrm{BMO}}^{1-q/p}, \tag{3.7}$$

which has been proven a number of times before; see Chen & Zhu (2005), Kozono & Wadade (2008), Dong & Xiao (2011) and Azzam & Bedrossian (2012). (In particular, one may adapt the elegant proof of (3.7) in Chen & Zhu (2005), which uses the John–Nirenberg inequality for functions in BMO, to give an elementary proof of (3.6) in the case $p = r$ (i.e. with just the L^p norm on the left-hand side) that bypasses the use of interpolation spaces; see McCormick et al. (2013).)

A number of related interpolation inequalities involving Besov spaces are proved in Bahouri et al. (2011). Of note is Theorem 1.43: for $s \in (0, n/2)$, we have

$$\|f\|_{L^p} \le c\|f\|^{1-2/p}_{\dot{B}^{s-n/2}_{\infty,\infty}}\|f\|^{2/p}_{\dot{H}^s},$$

with $p = \frac{2n}{n-2s}$. However, this does not include the endpoint $s = n/2$, and thus does not apply in our situation. Also of note is Theorem 2.42:

$$\|f\|_{L^p} \le c\|f\|^{1-2/p}_{\dot{B}^{-\alpha}_{\infty,\infty}}\|f\|^{2/p}_{\dot{H}^1} \qquad \text{for } \alpha = \frac{1}{p/2-1}.$$

In the case $p = 4$, this would imply (3.4) given the embedding $L^{2,\infty} \subset \dot{B}^{-1}_{\infty,\infty}$; however, we have not been able to find such an embedding in the literature.

3.1 Proof Using Fourier Transforms

3.1.1 A Weak-Strong Bernstein Inequality

Before embarking on our proof of the inequality

$$\|f\|_{L^p} \le c\|f\|^{\theta}_{L^{q,\infty}}\|f\|^{1-\theta}_{\dot{H}^s}$$

using Fourier transforms, we will require a result, known as Bernstein's inequality, that provides integrability of f assuming localisation of its Fourier transform: if $\hat{f}$ is supported in $B(0,R)$ (the ball of radius R) then for any $1 \le p \le q \le \infty$ if $f \in L^p(\mathbb{R}^n)$ then

$$\|f\|_{L^q} \le c_{p,q}R^{n(1/p-1/q)}\|f\|_{L^p}. \tag{3.8}$$

For our purposes we will require a version of this inequality that replaces L^p by $L^{p,\infty}$ on the right-hand side.

As in the standard proof of (3.8), we make use of the following simple result. We use the notation $\mathfrak{D}_h f(x) = h^{-n}f(x/h)$; note that $\widehat{\mathfrak{D}_h}(x) = \hat{f}(hx)$. Note that the *support* of $g \in \mathscr{S}'$ is the intersection of all closed sets K such that $\langle g, \phi\rangle = 0$ whenever the support of $\phi \in \mathscr{S}$ is disjoint from K.

Lemma 3.2. *There is a fixed $\phi \in \mathscr{S}$ such that if $\hat{f}$ is supported in $B(0,R)$ then $f = (\mathfrak{D}_{1/R}\phi) \star f$.*

Proof. Take $\phi \in \mathscr{S}$ so that $\hat{\phi} = 1$ on $B(0,1)$. Then

$$\widehat{\mathfrak{D}_{1/R}\phi}(\xi) = \hat{\phi}(\xi/R)$$

which is equal to 1 on $B(0,R)$. Thus $(\mathfrak{D}_{1/R}\phi) \star f - f$ has Fourier transform zero, and the lemma follows. $\square$

For use in the proof of our next lemma, note that

$$\|\mathfrak{D}_{1/R}\phi\|_{L^r} = R^{n(1-1/r)}\|\phi\|_{L^r}. \tag{3.9}$$

Lemma 3.3 (Weak-strong Bernstein inequality). *Let $1 < p < \infty$ and suppose that $f \in L^{p,\infty}(\mathbb{R}^n)$ and that $\hat{f}$ is supported in $B(0,R)$. Then for each q with $p < q < \infty$ there exists a constant $c_{p,q}$ such that*

$$\|f\|_{L^q} \leq cR^{n(1/p-1/q)}\|f\|_{L^{p,\infty}}. \tag{3.10}$$

Proof. We follow the standard proof, replacing Young's inequality by its weak form, and making use of the interpolation result of Lemma 2.2. First we prove the weak-weak version

$$\|f\|_{L^{q,\infty}} \leq cR^{n(1/p-1/q)}\|f\|_{L^{p,\infty}}$$

valid for all $1 < p \leq q < \infty$. To do this we simply apply the weak form of Young's inequality (Proposition 2.4) to $f = (\mathfrak{D}_{1/R}\phi) \star f$:

$$\begin{aligned}\|f\|_{L^{q,\infty}} &= \|(\mathfrak{D}_{1/R}\phi) \star f\|_{L^{q,\infty}} \\ &\leq c\|\mathfrak{D}_{1/R}\phi\|_{L^r}\|f\|_{L^{p,\infty}},\end{aligned}$$

where

$$1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$$

with $1 < p, q < \infty$ and $1 \leq r < \infty$. It follows using (3.9) that

$$\|f\|_{L^{s,\infty}} \leq cR^{n(1/p-1/s)}\|f\|_{L^{p,\infty}} \qquad \text{and} \qquad \|f\|_{L^{t,\infty}} \leq cR^{n(1/p-1/t)}\|f\|_{L^{p,\infty}}$$

for any $1 < p < s < q < t < \infty$. We then obtain (3.10) by interpolation of L^q between $L^{s,\infty}$ and $L^{t,\infty}$ (Lemma 2.2):

$$\begin{aligned}\|f\|_{L^q} &\leq c\|f\|_{L^{s,\infty}}^{\frac{s(t-q)}{q(t-s)}}\|f\|_{L^{t,\infty}}^{\frac{t(q-s)}{q(t-s)}} \\ &\leq cR^{n(1/p-1/q)}\|f\|_{L^{p,\infty}}.\end{aligned}$$

$\square$

3.1.2 Generalised Gagliardo–Nirenberg Inequality

Theorem 3.4. *Take $1 < q < p$ and $s \geq 0$ with $s > n(1/2 - 1/p)$. There exists a constant $c_{p,q,s}$ such that if $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ then $f \in L^p(\mathbb{R}^n)$ and*

$$\|f\|_{L^p} \leq c_{p,q,s} \|f\|_{L^{q,\infty}}^{\theta} \|f\|_{\dot{H}^s}^{1-\theta} \qquad \text{for every } f \in L^{q,\infty} \cap \dot{H}^s, \tag{3.11}$$

where

$$\frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right). \tag{3.12}$$

When $n(1/2 - 1/p) < s < n/2$, this theorem follows from interpolation using Lemma 2.2 coupled with the Sobolev embedding $\dot{H}^{n(1/2-1/p)} \hookrightarrow L^p$ (Theorem 2.6). However, the proof below applies to all $s > n(1/2 - 1/p)$; in particular the case we are most interested in is $s = n/2$, where the embedding $\dot{H}^{n/2} \hookrightarrow L^\infty$ just fails.

Proof. First we prove the theorem in the case $p \geq 2$. As in the proof of Theorem 2.6 we write

$$f = f_{<R} + f_{>R},$$

where $f_{<R}$ and $f_{>R}$ are defined in (2.15).

Using the endpoint Sobolev embedding $\dot{H}^{n(1/2-1/p)}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ from Theorem 2.6 (taking $\dot{H}^0 = L^2$ when $p = 2$) we can estimate

$$\begin{aligned}
\|f_{>R}\|_{L^p} &\leq c\|f_{>R}\|_{\dot{H}^{n(1/2-1/p)}} \\
&= c\left(\int_{|\xi|\geq R} |\xi|^{2n(1/2-1/p)} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2} \\
&\leq \frac{c}{R^{s-n(1/2-1/p)}} \left(\int_{|\xi|\geq R} |\xi|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2} \\
&= \frac{c}{R^{s-n(1/2-1/p)}} \|f\|_{\dot{H}^s},
\end{aligned}$$

while

$$\|f_{<R}\|_{L^p} \leq cR^{n(1/q-1/p)} \|f_{<R}\|_{L^{q,\infty}} \leq cR^{n(1/q-1/p)} \|f\|_{L^{q,\infty}}$$

using the weak-strong Bernstein inequality from Lemma 3.3 and (2.2). Thus

$$\|f\|_{L^p} \leq c(R^{n(1/q-1/p)} \|f\|_{L^{q,\infty}} + R^{-s+n(1/2-1/p)} \|f\|_{\dot{H}^s}).$$

Choosing

$$R^{s+n(1/q-1/2)} = \frac{\|f\|_{\dot{H}^s}}{\|f\|_{L^{q,\infty}}}$$

we obtain

$$\|f\|_{L^p} \le c\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot H^s}^{1-\theta}, \tag{3.13}$$

where

$$\theta = 1 - n\frac{1/q - 1/p}{s + n(1/q - 1/2)},$$

which on rearrangement yields the condition (3.12).

If $1 < q < p < 2$ then we first interpolate L^p between $L^{q,\infty}$ and L^2, and then use the above result with $p = 2$. Setting $\frac{1}{2} = \frac{\theta'}{q} + (1-\theta')\left(\frac{1}{2} - \frac{s}{n}\right)$ we have

$$\begin{aligned}
\|f\|_{L^p} &\le c\|f\|_{L^{q,\infty}}^{q(2-p)/p(2-q)}\|f\|_{L^2}^{2(p-q)/p(2-q)} \\
&\le c\|f\|_{L^{q,\infty}}^{q(2-p)/p(2-q)}\left(c\|f\|_{L^{q,\infty}}^{\theta'}\|f\|_{\dot H^s}^{1-\theta'}\right)^{2(p-q)/p(2-q)} \\
&= c\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot H^s}^{1-\theta},
\end{aligned}$$

with θ given by (3.12), as required. □

This proof is easily carried over to functions defined on $\mathbb{T}^n$ (that is, $[0,1]^n$ with periodic boundary conditions), using Fourier series. For simplicity we restrict our attention to (3.4) (i.e. the case $p = 4$, $q = 2$, $s = 1$): by analogy with Lemma 3.3 one can prove that

$$f = \sum_{|k|\le\kappa} \hat f_k e^{2\pi i k\cdot x} \qquad \Longrightarrow \qquad \|f\|_{L^4} \le c\kappa^{1/2}\|f\|_{L^{2,\infty}}. \tag{3.14}$$

Then, writing

$$f = \sum_{|k|\le\kappa} \hat f_k e^{2\pi i k\cdot x} + \sum_{|k|>\kappa} \hat f_k e^{2\pi i k\cdot x}$$

and using (3.14) and the Sobolev embedding $L^4 \subset \dot H^{1/2}$, we obtain

$$\begin{aligned}
\|f\|_{L^4} &\le c\kappa^{1/2}\|f\|_{L^{2,\infty}} + c\left(\sum_{|k|>\kappa} |k||f_k|^2\right)^{1/2} \\
&\le c\kappa^{1/2}\|f\|_{L^{2,\infty}} + c\kappa^{-1/2}\left(\sum_{|k|>\kappa} |k|^2|f_k|^2\right)^{1/2} \\
&\le c\kappa^{1/2}\|f\|_{L^{2,\infty}} + c\kappa^{-1/2}\|\nabla f\|_{L^2}.
\end{aligned}$$

Minimising over κ we obtain (3.4).

3.2 Proof Using Interpolation Spaces

For our second proof of the weak version of Ladyzhenskaya's inequality, we will use some of the standard theory of interpolation spaces. We recall here the basic facts we require: for full details, see the books of Bennett & Sharpley (1988), §5.1, and Bergh & Löfström (1976), §3.1.

Let (X_0, X_1) be a compatible couple of Banach spaces (that is, there is a Hausdorff topological vector space $\mathfrak{X}$ such that X_0 and X_1 embed continuously into $\mathfrak{X}$). The K-*functional* is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f,t) = K(f,t;X_0,X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\}$$

where the infimum is taken over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

Suppose $0 < \theta < 1$ and $1 \le q < \infty$, or $0 \le \theta \le 1$ and $q = \infty$. We define the interpolation space $(X_0, X_1)_{\theta,q}$ as the space of all $f \in X_0 + X_1$ for which the functional

$$\|f\|_{\theta,q} := \begin{cases} \left(\int_0^\infty [t^{-\theta} K(f,t)]^q \right)^{1/q} & 1 \le q < \infty \\ \sup_{0<t<\infty} t^{-\theta} K(f,t) & q = \infty \end{cases}$$

is finite. A very useful property of interpolation spaces is the estimate on the norms:

$$\|f\|_{\theta,q} \le c\|f\|_{X_0}^{1-\theta}\|f\|_{X_1}^{\theta} \tag{3.15}$$

(see Bergh & Löfström (1976), §3.5, p. 49). As a simple example of interpolation, note that

$$(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1-1/p,q} = L^{p,q}(\mathbb{R}^n)$$

if $1 < p < \infty$ and $1 \le q \le \infty$ (see Bennett & Sharpley (1988), Chapter 5, Theorem 1.9). In fact, this equality remains true with L^∞ replaced with BMO (see Bennett & Sharpley (1988), Chapter 5, Theorem 8.11):

$$(L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1-1/p,q} = L^{p,q}(\mathbb{R}^n). \tag{3.16}$$

The so-called *reiteration theorem* allows us to interpolate between interpolation spaces: it says that when we interpolate between two interpolation spaces of the same couple (X_0, X_1), we get another interpolation space in the same family.

Theorem 3.5 (Reiteration Theorem)**.** *Let (X_0, X_1) be a compatible couple of Banach spaces, let $0 \le \theta_0 < \theta_1 \le 1$, and let $1 \le q_0, q_1 \le \infty$. Set $A_0 = (X_0, X_1)_{\theta_0,q_0}$*

and $A_1 = (X_0, X_1)_{\theta_1,q_1}$. *If* $0 < \theta < 1$ *and* $1 \le q \le \infty$, *then*

$$(A_0, A_1)_{\theta,q} = (X_0, X_1)_{\theta',q}$$

providing $\theta' = (1-\theta)\theta_0 + \theta\theta_1$.

The proof may be found in Bennett & Sharpley (1988), Chapter 5, Theorem 2.4, or Bergh & Löfström (1976), Theorem 3.5.3.

3.2.1 Interpolation: Lorentz Version

Using the machinery of interpolation spaces, we can now give a very short proof of our generalised Ladyzhenskaya inequality (3.6).

Lemma 3.6 (Interpolation). *Let* $1 < q < p < \infty$, *and* $1 \le r \le \infty$. *For any* $f \in L^{q,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$,

$$\|f\|_{L^{p,r}} \le c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}.$$

Proof. Using (3.16), we have

$$L^{q,\infty}(\mathbb{R}^n) = (L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1-1/q,\infty}$$

provided that $1 < q < \infty$. Set $\mathfrak{B} := (L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1,\infty}$, and note that by (3.15) we have $\|f\|_{\mathfrak{B}} \le C\|f\|_{\mathrm{BMO}}$. By the Reiteration Theorem (Theorem 3.5), we obtain

$$L^{p,r}(\mathbb{R}^n) = (L^{q,\infty}(\mathbb{R}^n), \mathfrak{B})_{\alpha,r}$$

with $q < p < \infty$, provided that $\alpha = 1 - q/p$. Thus, using (3.15), we obtain

$$\|f\|_{L^{p,r}} \le c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathfrak{B}}^{1-q/p} \le c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p},$$

as required. □

Using the embedding $\dot{H}^1 \subset \mathrm{BMO}$ in two dimensions (see Section 2.3), for $f \in L^{2,\infty}(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)$, setting $n = 2$, $p = 4$ and $q = 2$ in Lemma 3.8 we obtain (3.4):

$$\|f\|_{L^4} \le c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|_{L^2}^{1/2}.$$

When Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, we may extend a function $f \in H_0^1(\Omega)$ by zero outside Ω and apply the above inequality on $\mathbb{R}^2$ to obtain the same for Ω.

3.2.2 Interpolation: Weak L^p Version

In the case $p = r$, we now give another proof of (3.6) which avoids the use of any Lorentz spaces (although it makes the proof a little more involved).

Lemma 3.7 (Weak interpolation)**.** *For any* $f \in L^{q,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$*, and any* $q < p < \infty$*,*

$$\|f\|_{L^{p,\infty}} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}.$$

Proof. Using (3.16), we have

$$L^{q,\infty}(\mathbb{R}^n) = (L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1-1/q,\infty}$$

provided that $1 < q < \infty$. Set $\mathfrak{B} := (L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1,\infty}$; note that by (3.15) we have $\|f\|_{\mathfrak{B}} \leq C\|f\|_{\mathrm{BMO}}$. By the Reiteration Theorem (Theorem 3.5), we obtain

$$L^{p,\infty}(\mathbb{R}^n) = (L^{q,\infty}(\mathbb{R}^n), \mathfrak{B})_{\alpha,\infty}$$

with $q < p < \infty$, provided that α solves $1 - \frac{1}{p} = (1-\alpha)(1-\frac{1}{q}) + \alpha \cdot 1$, or in other words that $\alpha = 1 - q/p$. Thus, using (3.15), we obtain

$$\|f\|_{L^{p,\infty}} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathfrak{B}}^{1-q/p} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p},$$

as required. □

By combining this with Lemma 2.2, we once again obtain (3.6).

Lemma 3.8 (Strong interpolation)**.** *For any* $f \in L^{q,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$*, and any* $q < p < \infty$*,*

$$\|f\|_{L^p} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}.$$

Proof. Given $p > q$, choose any r and s such that $q < r < p < s < \infty$. Then

$$\|f\|_{L^p} \leq c\|f\|_{L^{r,\infty}}^{1-\alpha}\|f\|_{L^{s,\infty}}^{\alpha},$$

where $\frac{1-\alpha}{r} + \frac{\alpha}{s} = \frac{1}{p}$. Applying Lemma 3.7 to the two factors on the right, we obtain

$$\begin{aligned}
\|f\|_{L^p} &\leq c(c\|f\|_{L^{q,\infty}}^{q/r}\|f\|_{\mathrm{BMO}}^{1-q/r})^{1-\alpha}(c\|f\|_{L^{q,\infty}}^{q/s}\|f\|_{\mathrm{BMO}}^{1-q/s})^{\alpha} \\
&\leq c\|f\|_{L^{q,\infty}}^{(1-\alpha)q/r+\alpha q/s}\|f\|_{\mathrm{BMO}}^{(1-\alpha)(1-q/r)+\alpha(1-q/s)} \\
&= c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p},
\end{aligned}$$

as required. □

Chapter 4

Existence and Uniqueness for Resistive Stokes-MHD

We consider now the Stokes-MHD equations

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.2a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.2b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.2c}$$

where $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$. The main purpose of this chapter is to prove the following theorem on the existence and uniqueness of weak solutions of (1.2) with $\nu, \eta > 0$.

First, let us define $\mathcal{D}_\sigma(\Omega) := \{\boldsymbol{u} \in C_c^\infty(\Omega) : \nabla\cdot\boldsymbol{u} = 0\}$ in the case when Ω is a bounded domain in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, and let $\mathcal{D}_\sigma(\mathbb{T}^n) := \{\boldsymbol{u} \in C^\infty(\mathbb{T}^n) : \nabla\cdot\boldsymbol{u} = 0\}$ (with the understanding that such $\boldsymbol{u}$ are periodic). Let $V(\Omega)$ be the closure of $\mathcal{D}_\sigma(\Omega)$ in the H^1 norm, let $H(\Omega)$ be the closure of $\mathcal{D}_\sigma(\Omega)$ in the L^2 norm, and finally let $V^*(\Omega)$ denote the dual of $V(\Omega)$.

Theorem 4.1. *Consider one of the following domains Ω:*

- $\Omega \subset \mathbb{R}^2$ *is a Lipschitz bounded domain;*
- $\Omega = \mathbb{R}^2$*; or*
- $\Omega = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Given an initial condition $\boldsymbol{B}_0 \in H(\Omega)$, for any $T > 0$ there exists a unique pair of *functions* $(\boldsymbol{u}, \boldsymbol{B})$ *with*

$$\begin{aligned}
\boldsymbol{u} &\in L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\boldsymbol{B} &\in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\frac{\partial \boldsymbol{B}}{\partial t} &\in L^2(0,T;V^*(\Omega)),
\end{aligned}$$

such that $\boldsymbol{B}(0) = \boldsymbol{B}_0$ *and for almost every* $t \in (0,T)$

$$\begin{aligned}
0 &= \nu\langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v}_1\rangle + \langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_1, \boldsymbol{B}\rangle,\\
\left\langle \frac{\partial \boldsymbol{B}}{\partial t}, \boldsymbol{v}_2 \right\rangle &= \eta\langle \nabla \boldsymbol{B}, \nabla \boldsymbol{v}_2\rangle + \langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_2, \boldsymbol{u}\rangle - \langle (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_2, \boldsymbol{B}\rangle,
\end{aligned}$$

for every pair of functions $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V(\Omega)$. *Furthermore, for any* $T > \varepsilon > 0$ *and any* $k \in \mathbb{N}$*,*

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(\varepsilon, T; H^k(\Omega)) \cap L^2(\varepsilon, T; H^{k+1}(\Omega)).$$

The proof of Theorem 4.1 is divided into several sections:

- In Section 4.1 we consider elliptic regularity for the Stokes equations

  $$\begin{aligned}
  -\nu\Delta \boldsymbol{u} + \nabla p &= \nabla\cdot \boldsymbol{f},\\
  \nabla\cdot\boldsymbol{u} &= 0,
  \end{aligned}$$

  in 2D, and show that $\boldsymbol{u} \in L^{2,\infty}$ whenever $\boldsymbol{f} \in L^1$.

- In Section 4.2, we use the results of the previous section and of Chapter 3 to prove global existence and uniqueness of (L^2-valued) weak solutions to (1.2) in a bounded domain $\Omega \subset \mathbb{R}^2$ and the whole of $\mathbb{R}^2$. (The proof for $\mathbb{T}^2$ is analogous, and we omit it.)

- In Section 4.3 we prove higher-order estimates to show that the solutions stay as smooth as the initial data permits for all time, and hence that after any arbitrary time $\varepsilon > 0$ the solution is smooth.

Finally, in Section 4.4, we outline the changes necessary to prove existence (but not uniqueness) of weak solutions to (1.2) in three dimensions, both in the case where $\Omega \subset \mathbb{R}^3$ is a bounded domain, and the whole space $\Omega = \mathbb{R}^3$.

Before we begin our formal treatment of the problem, it is instructive to note that (in the whole space case) the equations (1.2) are invariant under the rescaling

$$\boldsymbol{u}(x,t) \mapsto \lambda\boldsymbol{u}(\lambda x, \lambda^2 t), \qquad \boldsymbol{B}(x,t) \mapsto \lambda\boldsymbol{B}(\lambda x, \lambda^2 t).$$

In the two-dimensional case the critical (scale-invariant) spaces include the natural energy space L^2 in which we pose the problem for $\boldsymbol{B}$, and the space $L^{2,\infty}$ in which the corresponding velocity field $\boldsymbol{u}$ then lies (due to the elliptic nature of equation (1.2a), see Section 4.1).

As noted in the introduction, existence results for the three-dimensional Navier–Stokes equations in such critical spaces have received much attention in recent years, the standard technique being to recast the equations in integral form and seek the solution as the fixed point of the resulting integral operator in an appropriately chosen Banach space. However, for L^2-valued weak solutions (in 2D and 3D) of the kind we study here, it is more usual (and significantly simpler) to employ a proof based on the Galerkin method and relatively elementary energy estimates.

4.1 The Stokes Operator and Elliptic Regularity in L^1

We now consider the Stokes equation

$$-\nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\boldsymbol{f}, \tag{4.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4.1b}$$

on one of the three domains in Theorem 4.1, with Dirichlet boundary conditions if Ω is bounded. By setting $\boldsymbol{f} = \boldsymbol{B}\otimes\boldsymbol{B}$ (i.e. $\boldsymbol{f}_{i,j} = \boldsymbol{B}_i\boldsymbol{B}_j$) we recover equation (1.2a), since $\boldsymbol{B}$ is divergence-free. In this case, if $\boldsymbol{B}\in L^2(\Omega)$, then $\boldsymbol{B}\otimes\boldsymbol{B}$ is in $L^1(\Omega)$, so the right-hand side behaves like the derivative of an L^1 function. If $\boldsymbol{f}\in L^p(\Omega)$ for $p > 1$, one would expect that $\boldsymbol{u}\in W^{1,p}(\Omega)$, but this does not hold for $p = 1$. If it did, in two dimensions we would obtain $\boldsymbol{u}\in W^{1,1}(\Omega)\subset L^2(\Omega)$. In fact, in this section we prove that, when $\boldsymbol{f}\in L^1(\Omega)$ in (4.1), then $\boldsymbol{u}\in L^{2,\infty}(\Omega)$.

The solution of equation (4.1) is given by integration against the Green's function: let $\boldsymbol{U}$, q solve

$$\begin{aligned} -\nu\Delta\boldsymbol{U}(x,y) + \nabla q(x,y) &= \delta(x-y), \\ \nabla\cdot\boldsymbol{U} &= 0, \end{aligned}$$

where δ denotes the Dirac delta function. Then the solution of (4.1) is given by

$$\begin{aligned} \boldsymbol{u} &= \int_\Omega \boldsymbol{U}(x,y)(\nabla\cdot\boldsymbol{f}(y))\,\mathrm{d}y, \\ p &= \int_\Omega q(x,y)(\nabla\cdot\boldsymbol{f}(y))\,\mathrm{d}y. \end{aligned}$$

Integrating by parts with respect to k, we obtain

$$\begin{aligned}\boldsymbol{u}_i(x) = [\boldsymbol{U} * (\nabla \cdot \boldsymbol{f})]_i(x) &= \sum_{j=1}^{2} \int_{\mathbb{R}^2} \boldsymbol{U}_{i,j}(x-y) \sum_{k=1}^{2} \partial_k f_{k,j}(y)\,\mathrm{d}y \\ &= -\sum_{j,k=1}^{2} \int_{\mathbb{R}^2} \partial_k \boldsymbol{U}_{i,j}(x,y) f_{k,j}(y)\,\mathrm{d}y.\end{aligned}$$

So if $\partial_k \boldsymbol{U} \in L^{2,\infty}(\Omega)$, then by Young's inequality for convolutions we can show that $\|\boldsymbol{u}\|_{L^{2,\infty}} \leq C\|\boldsymbol{f}\|_{L^1}$.

In the case $\Omega = \mathbb{R}^2$, we have explicit formulae for $\boldsymbol{U}$ and q (see Galdi (2011), §IV.2): abusing notation, $\boldsymbol{U}_{i,j}(x,y) = \boldsymbol{U}_{i,j}(x-y)$ and $q_{i,j}(x,y) = q_{i,j}(x-y)$, where

$$\begin{aligned}\boldsymbol{U}_{i,j}(x) &= \frac{1}{4\pi\nu}\left[\frac{x_i x_j}{|x|^2} - \delta_{ij}\log|x|\right], \\ q_j(x) &= \frac{1}{2\pi}\frac{x_j}{|x|^2}.\end{aligned}$$

Now,

$$\partial_k \boldsymbol{U}_{i,j}(x) = \frac{1}{4\pi\nu}\left[\frac{\delta_{ik}x_j + \delta_{kj}x_i}{|x|^2} - \frac{x_i x_j x_k}{|x|^4} - \delta_{ij}\frac{x_k}{|x|^2}\right],$$

and so

$$|\partial_k \boldsymbol{U}_{i,j}(x)| \leq \frac{1}{\pi\nu|x|}.$$

As noted in Section 2.1.1, $\frac{1}{|x|}$ is in $L^{2,\infty}(\mathbb{R}^2)$, and $\|\partial_k \boldsymbol{U}_{i,j}\|_{L^{2,\infty}} \leq \frac{1}{\nu\sqrt{\pi}}$. Using Young's inequality (Proposition 2.4), we obtain

$$\|\boldsymbol{u}\|_{L^{2,\infty}} \leq c\|\partial_k \boldsymbol{U}_{i,j}\|_{L^{2,\infty}}\|\boldsymbol{f}\|_{L^1} \leq c\|\boldsymbol{f}\|_{L^1}. \tag{4.2}$$

Thus, whenever $\boldsymbol{f} \in L^1(\mathbb{R}^2)$, $\boldsymbol{u} \in L^{2,\infty}(\mathbb{R}^2)$.

In the case where $\Omega = \mathbb{T}^2$, one can also write down an explicit formula for the fundamental solution — see Hasimoto (1959) and Cichocki & Felderhof (1989), for example — and obtain (4.2) again; the details are very similar to the above case, and we omit them.

In the case where Ω is a bounded Lipschitz domain, while we no longer have an explicit formula for the Green's function $\boldsymbol{U}$, by Theorem 7.1 in Mitrea & Mitrea (2011) we have $\nabla\boldsymbol{U}(x,\cdot) \in L^{2,\infty}(\Omega)$ uniformly for $x \in \Omega$. Using a straightforward generalisation of Young's inequality to expressions of the form $\int_\Omega G(x,y)f(y)\,\mathrm{d}y$ when G is symmetric, we obtain (4.2) on a bounded Lipschitz domain as well; i.e. whenever $\boldsymbol{f} \in L^1(\Omega)$, $\boldsymbol{u} \in L^{2,\infty}(\Omega)$.

4.2 Existence and Uniqueness of Weak Solutions

We return now to the system

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.2a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.2b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0. \tag{1.2c}$$

We will show that system (1.2) has a unique weak solution for all time in the three cases of Ω described in Theorem 4.1. We first define a weak solution in line with the terminology used for the Navier–Stokes equations (see, e.g., Temam (2001)).

Definition 4.2. *A pair of functions* $(\boldsymbol{u},\boldsymbol{B})$ *is a* weak solution *of* (1.2) *on* $(0,T)$ *if*

$$\begin{aligned}
\boldsymbol{u} &\in L^\infty(0,T;L^{n/(n-1),\infty}(\Omega))\cap L^2(0,T;V(\Omega)),\\
\boldsymbol{B} &\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;V(\Omega)),\\
\frac{\partial\boldsymbol{B}}{\partial t} &\in L^1(0,T;V^*(\Omega)),
\end{aligned}$$

such that for almost every $t\in(0,T)$

$$\begin{aligned}
0 &= \nu\langle\nabla\boldsymbol{u},\nabla\boldsymbol{v}_1\rangle + \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_1,\boldsymbol{B}\rangle,\\
\left\langle\frac{\partial\boldsymbol{B}}{\partial t},\boldsymbol{v}_2\right\rangle &= \eta\langle\nabla\boldsymbol{B},\nabla\boldsymbol{v}_2\rangle + \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_2,\boldsymbol{u}\rangle - \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_2,\boldsymbol{B}\rangle,
\end{aligned}$$

for every pair of functions $\boldsymbol{v}_1,\boldsymbol{v}_2\in V(\Omega)$.

Note that the pressure p is uniquely determined up to an additive function of time by $\boldsymbol{u}$ and $\boldsymbol{B}$ by solution of an elliptic boundary value problem; see the discussion around equation (4.4) and the books Chemin et al. (2006) and Foias et al. (2001).

In this section we will prove the following theorem.

Theorem 4.3. *Given* $\boldsymbol{B}_0\in H(\Omega)$*, for any* $T>0$ *there exists a unique weak solution* $(\boldsymbol{u},\boldsymbol{B})$ *of* (1.2) *on* $(0,T)$*, such that* $\frac{\partial\boldsymbol{B}}{\partial t}\in L^2(0,T;V^*(\Omega))$ *and* $\boldsymbol{B}\in C^0([0,T];L^2(\Omega))$*, with* $\boldsymbol{B}(0)=\boldsymbol{B}_0$.

In Section 4.2.1, we prove existence of a weak solution in the case $\Omega\subset\mathbb{R}^2$ is a Lipschitz bounded domain, while in Section 4.2.2 we prove existence of a weak solution in the case $\Omega=\mathbb{R}^2$. The proof of existence in the case where $\Omega=\mathbb{T}^2$ is similar to the previous two, and we omit it. Finally, in Section 4.2.3, we prove uniqueness of weak solutions.

4.2.1 Global Existence of Solutions in a Bounded Domain

In this subsection we prove existence of a weak solution on a Lipschitz bounded domain $\Omega \subset \mathbb{R}^2$, with Dirichlet boundary conditions. As with the 2D Navier–Stokes equations, we use energy methods and Galerkin approximations. To do so, we first set up some notation.

Let Π be the Leray projection $\Pi\colon L^2(\Omega) \to H$, i.e. the orthogonal projection from L^2 onto H. We define the *Stokes operator* as $A := -\Pi\Delta$. Let $\{\phi_m\}_{m\in\mathbb{N}} \subset C^\infty(\Omega)$ be the collection of eigenfunctions of the Stokes operator on Ω with Dirichlet boundary conditions, ordered such that the eigenvalues associated to ϕ_m are non-decreasing with respect to m. Let V_m be the subspace of H spanned by $\phi_1, \dots, \phi_m$, and let $P_m\colon H \to V_m$ be the orthogonal projection onto V_m.

In order to use the Galerkin method, we consider the equations

$$-\nu\Delta\boldsymbol{u}^m + \nabla p_*^m = (\boldsymbol{B}^m \cdot \nabla)\boldsymbol{B}^m, \tag{4.3a}$$

$$\frac{\partial \boldsymbol{B}^m}{\partial t} + P_m[(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{B}^m] - \eta\Delta\boldsymbol{B}^m = P_m[(\boldsymbol{B}^m \cdot \nabla)\boldsymbol{u}^m], \tag{4.3b}$$

$$\nabla \cdot \boldsymbol{u}^m = \nabla \cdot \boldsymbol{B}^m = 0. \tag{4.3c}$$

Note that we do not insert a P_m on the right-hand side of (4.3a): this will make some of our convergence arguments easier (see Proposition 4.6).

Thinking of $\boldsymbol{u}^m$ as a function of $\boldsymbol{B}^m$ given by equation (4.3a), it is easy to check that (4.3b) is a locally Lipschitz ODE on the finite-dimensional space V_m, and thus by existence and uniqueness theory for finite-dimensional ODEs (Picard's theorem), there exists a unique solution $\boldsymbol{B}^m \in V_m$ of equation (4.3b), with $\boldsymbol{u}^m$ given by equation (4.3a).

Proposition 4.4 (Energy estimates)**.** *The Galerkin approximations are uniformly bounded in the following senses:*

$$\begin{aligned} \boldsymbol{u}^m &\ \textit{is uniformly bounded in } L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)), \\ \boldsymbol{B}^m &\ \textit{is uniformly bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega)), \\ \frac{\partial \boldsymbol{B}^m}{\partial t} &\ \textit{is uniformly bounded in } L^2(0,T;V^*(\Omega)). \end{aligned}$$

Proof. Take the inner product of equation (4.3a) with $\boldsymbol{u}^m$ and the inner product of equation (4.3b) with $\boldsymbol{B}^m$, and add to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}^m(t)\|_{L^2}^2 + \nu\|\nabla\boldsymbol{u}^m(t)\|_{L^2}^2 + \eta\|\nabla\boldsymbol{B}^m(t)\|_{L^2}^2 = 0.$$

Integrating over $[0,t]$ we obtain

$$\|\boldsymbol{B}^m(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla \boldsymbol{u}^m(s)\|_{L^2}^2\,\mathrm{d}s + 2\eta\int_0^t \|\nabla \boldsymbol{B}^m(s)\|_{L^2}^2\,\mathrm{d}s \\ = \|\boldsymbol{B}^m(0)\|_{L^2}^2 \leq \|\boldsymbol{B}_0\|_{L^2}^2,$$

so $\boldsymbol{u}^m \in L^2(0,T;V(\Omega))$ and $\boldsymbol{B}^m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega))$.

As in Section 4.1, the solution $\boldsymbol{u}^m$ to equation (4.3a) is given by convolution with $\boldsymbol{U}$, the Green's function for the Stokes equations. By (4.2), we have

$$\|\boldsymbol{u}^m(t)\|_{L^{2,\infty}} \leq c\|(\boldsymbol{B}^m(t))^2\|_{L^1} \leq c\|\boldsymbol{B}^m(t)\|_{L^2}^2,$$

so $\boldsymbol{u}^m \in L^\infty(0,T;L^{2,\infty}(\Omega))$.

For the estimate on $\frac{\partial \boldsymbol{B}^m}{\partial t}$, taking the norm in V^* of the $\boldsymbol{B}$ equation yields

$$\left\|\frac{\partial \boldsymbol{B}^m}{\partial t}\right\|_{V^*} \leq \eta\|\boldsymbol{B}^m\|_V + \|P_m[(\boldsymbol{B}^m\cdot\nabla)\boldsymbol{u}^m]\|_{V^*} + \|P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]\|_{V^*}.$$

For $\phi \in V(\Omega)$,

$$|\langle P_m[(\boldsymbol{B}^m\cdot\nabla)\boldsymbol{u}^m],\phi\rangle| \leq \|\boldsymbol{B}^m\|_{L^4}\|\boldsymbol{u}^m\|_{L^4}\|\nabla\phi\|_{L^2},$$

so $\|P_m[(\boldsymbol{B}^m\cdot\nabla)\boldsymbol{u}^m]\|_{V^*} \leq \|\boldsymbol{B}^m\|_{L^4}\|\boldsymbol{u}^m\|_{L^4}$ (and the same for the other term). By applying Ladyzhenskaya's inequality (3.1) to $\boldsymbol{B}^m$ and our weak Ladyzhenskaya's inequality (3.4) to $\boldsymbol{u}^m$, we obtain the the following estimate:

$$\left\|\frac{\partial \boldsymbol{B}^m}{\partial t}\right\|_{V^*}^2 \leq \eta\|\boldsymbol{B}^m\|_V^2 + c\|\boldsymbol{B}^m\|_{L^2}\|\boldsymbol{B}^m\|_V\|\boldsymbol{u}^m\|_{L^{2,\infty}}\|\boldsymbol{u}^m\|_V,$$

as required. □

We use the Banach–Alaoglu theorem to extract a convergent subsequence, which we relabel as $\boldsymbol{B}^m$, such that

$$\begin{aligned} \boldsymbol{B}^m &\overset{*}{\rightharpoonup} \boldsymbol{B} && \text{in } L^\infty(0,T;L^2(\Omega)),\\ \boldsymbol{B}^m &\rightharpoonup \boldsymbol{B} && \text{in } L^2(0,T;V(\Omega)),\\ \frac{\partial \boldsymbol{B}^m}{\partial t} &\overset{*}{\rightharpoonup} \frac{\partial \boldsymbol{B}}{\partial t} && \text{in } L^2(0,T;V^*(\Omega)). \end{aligned}$$

This weak convergence is not enough to guarantee the convergence of the nonlinear terms; we need strong convergence, which we obtain by means of the Aubin–Lions compactness lemma.

Theorem 4.5 (Aubin–Lions compactness lemma)**.** *Let $X \subset B \subset Y$ be Banach spaces such that the inclusion $X \Subset B$ is a compact embedding. Then, for any $1 < p < \infty$ and any $1 \leq q < \infty$, the space*

$$\left\{ f : f \in L^p(0,T;X) \text{ and } \frac{\partial f}{\partial t} \in L^q(0,T;Y) \right\}$$

is compactly embedded in $L^p(0,T;B)$.

Proof. The original result of Aubin (1963) and Lions (1969) covers the case when $1 < p, q < \infty$. Chapter 3 of Temam (2001) contains both the original case (see Theorem 2.1, p. 185), as well as the case $p = 2$, $q = 1$, whenever X and Y are Hilbert spaces (see Theorem 2.3, p. 187). The general case (and many other similar results) is proved in the paper of Simon (1987), §8, Theorem 5 and Corollary 4. $\square$

By the Aubin–Lions compactness lemma, we may extract a subsequence such that $\boldsymbol{B}^m \to \boldsymbol{B}$ strongly in $L^2(0,T;L^2(\Omega))$ and strongly in $C^0([0,T];V^*(\Omega))$. In particular, this gives sense to the initial data with $\lim_{t\to 0^+} \boldsymbol{B}(t) = \boldsymbol{B}_0$ as a limit in $V^*(\Omega)$.

Since the limit $\boldsymbol{B} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega))$, it is straightforward to show that $(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} \in L^2(0,T;V^*(\Omega))$. This allows us to *define* $\boldsymbol{u}$ to be the unique solution of

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{4.4a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4.4b}$$

where $\boldsymbol{u} \in L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega))$ and $p_* \in L^2(0,T;L^2(\Omega))$ is given by standard elliptic theory for the Stokes equations (see Section 4.1 above, and Lemma 2.1 in Chapter 1 of Temam (2001)).

We now want to show that $\boldsymbol{u}^m$ does indeed converge to $\boldsymbol{u}$ in the appropriate senses; this will allow us to show that the nonlinear terms involving $\boldsymbol{u}$ converge and thus that the $\boldsymbol{B}$ equation is satisfied in the limit.

Proposition 4.6. *The Galerkin approximations $\boldsymbol{u}^m$ converge to $\boldsymbol{u}$ strongly in $L^2(0,T;L^{2,\infty}(\Omega))$, and weakly in $L^2(0,T;V(\Omega))$.*

Proof. Subtracting the equations for $\boldsymbol{u}^m$ and $\boldsymbol{u}$, we obtain

$$\begin{aligned}-\nu\Delta(\boldsymbol{u}^m - \boldsymbol{u}) + \nabla(p_*^m - p_*) &= \nabla\cdot(\boldsymbol{B}^m\otimes\boldsymbol{B}^m - \boldsymbol{B}\otimes\boldsymbol{B}) \\ &= \nabla\cdot[\boldsymbol{B}^m\otimes(\boldsymbol{B}^m - \boldsymbol{B}) + (\boldsymbol{B}^m - \boldsymbol{B})\otimes\boldsymbol{B}].\end{aligned}$$

By elliptic regularity from Section 4.1, we obtain

$$\begin{aligned}\|\boldsymbol{u}^m - \boldsymbol{u}\|_{L^{2,\infty}} &\le c\|\boldsymbol{B}^m \otimes (\boldsymbol{B}^m - \boldsymbol{B})\|_{L^1} + c\|(\boldsymbol{B}^m - \boldsymbol{B}) \otimes \boldsymbol{B}\|_{L^1} \\ &\le c\|\boldsymbol{B}^m - \boldsymbol{B}\|_{L^2} (\|\boldsymbol{B}^m\|_{L^2} + \|\boldsymbol{B}\|_{L^2}) \\ &\le c(K+M)\|\boldsymbol{B}^m - \boldsymbol{B}\|_{L^2},\end{aligned}$$

where $K = \sup_{m\in\mathbb{N}} \sup_{t\in[0,T]} \|\boldsymbol{B}^m\|_{L^2}$, $M = \sup_{t\in[0,T]} \|\boldsymbol{B}\|_{L^2}$. Squaring and integrating in time yields

$$\int_0^T \|\boldsymbol{u}^m(t) - \boldsymbol{u}(t)\|_{L^{2,\infty}}^2 \,\mathrm{d}t \le c \int_0^T \|\boldsymbol{B}^m(t) - \boldsymbol{B}(t)\|_{L^2}^2 \,\mathrm{d}t.$$

As the right-hand side converges to zero, so does the left-hand side, and hence $\boldsymbol{u}^m \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{2,\infty}(\Omega))$. Let $\boldsymbol{v}$ be the weak limit of $\boldsymbol{u}^m$ in $L^2(0,T;V(\Omega))$; it remains to show that $\boldsymbol{u} = \boldsymbol{v}$. As $V(\Omega) \subset L^2(\Omega) \subset L^{2,\infty}(\Omega)$, we have $(L^{2,\infty})^*(\Omega) \subset L^2(\Omega) \subset V^*(\Omega)$. So if $\boldsymbol{u}^m \rightharpoonup \boldsymbol{v}$ in $L^2(0,T;V(\Omega))$, then $\boldsymbol{u}^m \rightharpoonup \boldsymbol{v}$ in $L^2(0,T;L^{2,\infty}(\Omega))$ (because we are testing with a smaller set of functionals). But $\boldsymbol{u}^m \to \boldsymbol{u}$ strongly (and hence also weakly) in $L^2(0,T;L^{2,\infty}(\Omega))$, and thus by uniqueness of weak limits $\boldsymbol{u} = \boldsymbol{v}$, and the proposition is proved. $\square$

We now proceed to show that the nonlinear terms in the $\boldsymbol{B}$ equation converge. The following proposition is symmetric in $\boldsymbol{B}$ and $\boldsymbol{u}$, and thus applies to both the $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ and $(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}$ terms.

Proposition 4.7. *Suppose that:*

- $\boldsymbol{u}^m \to \boldsymbol{u}$ *and* $\boldsymbol{B}^m \to \boldsymbol{B}$ *(strongly) in* $L^2(0,T;L^{2,\infty}(\Omega))$*; and*
- $\boldsymbol{u}^m$, $\boldsymbol{B}^m$ *are uniformly bounded in* $L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega))$.

Then after passing to a subsequence $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m] \overset{*}{\rightharpoonup} (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ *in* $L^2(0,T;V^*(\Omega))$.

Proof. For $\phi \in V(\Omega)$,

$$|\langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m], \phi\rangle| \le \|\boldsymbol{u}^m\|_{L^4}\|\boldsymbol{B}^m\|_{L^4}\|\nabla\phi\|_{L^2},$$

so by the weak Ladyzhenskaya inequality (3.4),

$$\|P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]\|_{V^*} \le c\|\boldsymbol{u}^m\|_{L^{2,\infty}}^{1/2}\|\nabla\boldsymbol{u}^m\|_{L^2}^{1/2}\|\boldsymbol{B}^m\|_{L^{2,\infty}}^{1/2}\|\nabla\boldsymbol{B}^m\|_{L^2}^{1/2}.$$

Hence $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]$ are uniformly bounded in $L^2(0,T;V^*(\Omega))$. Therefore a subsequence of $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]$ converges weakly-$*$ in $L^2(0,T;V^*(\Omega))$; as usual we relabel this subsequence as the original sequence.

To show that the limit is indeed $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$, we test with a slightly more regular test function. Let $\phi \in C^0([0,T];V(\Omega))$. Then

$$\begin{aligned}
&\int_0^T \langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m] - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, \phi\rangle \,\mathrm{d}t \\
&= \underbrace{\int_0^T \langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}], \phi\rangle \,\mathrm{d}t}_{\mathrm{I}} + \underbrace{\int_0^T \langle (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, P_m\phi - \phi\rangle \,\mathrm{d}t}_{\mathrm{II}}\,.
\end{aligned}$$

Clearly II converges since $P_m\phi \to \phi$ in $L^2(0,T;V(\Omega))$. For the first integral, we have

$$\begin{aligned}
\mathrm{I} &= \int_0^T \langle (\boldsymbol{u}^m\cdot\nabla)(\boldsymbol{B}^m - \boldsymbol{B}) + ((\boldsymbol{u}^m - \boldsymbol{u})\cdot\nabla)\boldsymbol{B}, P_m\phi\rangle \,\mathrm{d}t \\
&\leq \max_{t\in[0,T]} \|\nabla\phi\|_{L^2} \int_0^T \left(\|\boldsymbol{u}^m\|_{L^4}\|\boldsymbol{B}^m - \boldsymbol{B}\|_{L^4} + \|\boldsymbol{u}^m - \boldsymbol{u}\|_{L^4}\|\boldsymbol{B}\|_{L^4}\right) \mathrm{d}t.
\end{aligned}$$

By the weak Ladyzhenskaya inequality (3.4), and the fact that $\boldsymbol{u}^m \to \boldsymbol{u}$ and $\boldsymbol{B}^m \to \boldsymbol{B}$ in $L^2(0,T;L^{2,\infty}(\Omega))$, the right-hand side of the above expression tends to zero. Thus

$$\int_0^T \langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m] - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, \phi\rangle \,\mathrm{d}t \to 0 \quad \text{for all } \phi \in C^0([0,T];V(\Omega)),$$

and therefore $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m] \overset{*}{\rightharpoonup} (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ in $L^2(0,T;V^*(\Omega))$ by uniqueness of weak-$*$ limits. □

Hence $(\boldsymbol{u},\boldsymbol{B})$ is indeed a weak solution of (1.2). Since $\boldsymbol{B} \in L^2(0,T;V(\Omega))$ and $\frac{\partial \boldsymbol{B}}{\partial t} \in L^2(0,T;V^*(\Omega))$, it follows that $\boldsymbol{B} \in C^0([0,T];L^2(\Omega))$ (see Evans (1998), §5.9.2, Theorem 3). Since we already know that $\lim_{t\to 0^+}\boldsymbol{B}(t) = \boldsymbol{B}_0$ as a limit in $V^*(\Omega)$, it follows that $\lim_{t\to 0^+}\boldsymbol{B}(t) = \boldsymbol{B}_0$ as a limit in $L^2(\Omega)$. This completes the proof of Theorem 4.3 in the case where Ω is a Lipschitz bounded domain in $\mathbb{R}^2$.

4.2.2 Global Existence of Weak Solutions in $\mathbb{R}^2$

We turn now to the proof of Theorem 4.3 in the case $\Omega = \mathbb{R}^2$. We apply Fourier truncations to the equations, and then show convergence as $R \to \infty$. The arguments are not so different from those in the previous section, so we only outline the main changes.

Define the Fourier truncation $\mathcal{S}_R$ by $\widehat{\mathcal{S}_R f}(\xi) = \mathbb{1}_{B_R}(\xi)\hat{f}(\xi)$, where B_R denotes the ball of radius R centered at the origin. We consider the truncated MHD equations on the whole of $\mathbb{R}^2$ as follows:

$$-\nu\Delta \boldsymbol{u}^R + \nabla p_*^R = (\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R, \tag{4.5a}$$

$$\frac{\partial \boldsymbol{B}^R}{\partial t} + \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R] - \Delta \boldsymbol{B}^R = \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R], \tag{4.5b}$$

$$\nabla \cdot \boldsymbol{u}^R = \nabla \cdot \boldsymbol{B}^R = 0, \tag{4.5c}$$

with initial data $\mathcal{S}_R\boldsymbol{B}_0$. Note that, as before, there is no $\mathcal{S}_R$ on the right-hand side of (4.5a). By taking the cutoff initial data as we have, we ensure that, for $t \geq 0$, $\boldsymbol{B}^R$ lies in the space

$$V_R := \{f \in L^2(\mathbb{R}^n) : \hat{f} \text{ is supported in } B_R\},$$

as the truncations are invariant under the flow of the equations; this implies that $\boldsymbol{u}_R \in V_{2R}$. The Fourier truncations act like mollifiers, smoothing the equation; in particular, it is easy to show that

$$F(\boldsymbol{u}^R, \boldsymbol{B}^R) := \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R]$$

is Lipschitz as a map $F\colon V_{2R} \times V_R \to V_R$. Therefore $\frac{\partial \boldsymbol{B}^R}{\partial t} = G(\boldsymbol{B}^R)$ for some Lipschitz function $G\colon V_R \to V_R$, so by Picard's theorem for infinite-dimensional ODEs, equation (4.5b) will have a unique solution $\boldsymbol{B}^R \in V_R$, and $\boldsymbol{u}^R \in V_{2R}$ is given by equation (4.5a).

Repeating the estimates of Proposition 4.4, with slight modifications to account for the truncations, we again have the following:

$$\begin{aligned} \boldsymbol{u}^R &\text{ is uniformly bounded in } L^\infty(0,T;L^{2,\infty}(\mathbb{R}^2)) \cap L^2(0,T;V(\mathbb{R}^2)),\\ \boldsymbol{B}^R &\text{ is uniformly bounded in } L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;V(\mathbb{R}^2)),\\ \frac{\partial \boldsymbol{B}^R}{\partial t} &\text{ is uniformly bounded in } L^2(0,T;V^*(\mathbb{R}^2)). \end{aligned}$$

Because we are working on the whole of $\mathbb{R}^2$, we cannot apply the Aubin–Lions compactness lemma directly (because the embedding $H^1 \subset L^2$ is no longer compact). Instead, there exists a subsequence of $\boldsymbol{B}^R$ that converges strongly in $L^2(0,T;L^2(K))$ for any compact subset $K \subset \mathbb{R}^2$ (see Proposition 2.7 in Chemin et al. (2006)), and the limit satisfies

$$\boldsymbol{B} \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;V(\mathbb{R}^2)).$$

Thus, we may again define $\boldsymbol{u}$ to be the unique solution of equation (4.4). We now show that $\boldsymbol{u}^R$ converges strongly to $\boldsymbol{u}$ in $L^2(0,T;L^{2,\infty}(K))$ for any compact subset $K \subset \mathbb{R}^2$. This is a little more delicate than the previous case, dealt with in Proposition 4.6: $\boldsymbol{u}$ depends on $\boldsymbol{B}$ on the whole space, but $\boldsymbol{B}^R$ only converges strongly on compact subsets, so we must take care to derive the strong convergence of $\boldsymbol{u}^R$.

Proposition 4.8. *For any compact subset $K \subset \mathbb{R}^2$, the Fourier truncations $\boldsymbol{u}^R$ converge to $\boldsymbol{u}$ strongly in $L^2(0,T;L^{2,\infty}(K))$.*

Proof. It suffices to consider $K = B_r$ for any $r > 0$. Set $\mathfrak{B}^R := \boldsymbol{B}^R \otimes \boldsymbol{B}^R$ and $\mathfrak{B} := \boldsymbol{B} \otimes \boldsymbol{B}$. Since $\boldsymbol{B}^R, \boldsymbol{B} \in L^\infty(0,T;L^2(\mathbb{R}^2))$, $\mathfrak{B}^R, \mathfrak{B} \in L^\infty(0,T;L^1(\mathbb{R}^2))$. Moreover, since also $\boldsymbol{B}^R, \boldsymbol{B} \in L^2(0,T;\dot{H}^1(\mathbb{R}^2))$,

$$\|\partial_k(\boldsymbol{B} \otimes \boldsymbol{B})\|_{L^1} = 2\|\boldsymbol{B} \otimes (\partial_k \boldsymbol{B})\|_{L^1} \leq 2\|\boldsymbol{B}\|_{L^2}\|\nabla \boldsymbol{B}\|_{L^2}$$

and since the right-hand side is L^2 in time, $\mathfrak{B}^R, \mathfrak{B} \in L^2(0,T;\dot{W}^{1,1}(\mathbb{R}^2))$. Because $\dot{W}^{1,1}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$, we get

$$\mathfrak{B}^R, \mathfrak{B} \in L^\infty(0,T;L^1(\mathbb{R}^2)) \cap L^2(0,T;L^2(\mathbb{R}^2)).$$

Since $\boldsymbol{B}^R \to \boldsymbol{B}$ strongly in $L^2(0,T;L^2(B_r))$,

$$\begin{aligned}\|\mathfrak{B}^R - \mathfrak{B}\|_{L^1(B_r)} &\leq \|\boldsymbol{B}^R \otimes (\boldsymbol{B}^R - \boldsymbol{B})\|_{L^1(B_r)} + \|(\boldsymbol{B}^R - \boldsymbol{B}) \otimes \boldsymbol{B}\|_{L^1(B_r)} \\ &\leq M\|\boldsymbol{B}^R - \boldsymbol{B}\|_{L^2(B_r)},\end{aligned}$$

where $\|\boldsymbol{B}^R\|_{L^2(\mathbb{R}^2)}, \|\boldsymbol{B}\|_{L^2(\mathbb{R}^2)} \leq M$ for all time, and hence $\mathfrak{B}^R \to \mathfrak{B}$ strongly in $L^2(0,T;L^1(B_r))$.

Let $G = \partial_k \boldsymbol{U}$ be the derivative of the fundamental solution of the Stokes equation (see Section 4.1). Then

$$\begin{aligned}\boldsymbol{u}^R(x) - \boldsymbol{u}(x) &= \int_{|y| \leq M+r} G(x-y)[\mathfrak{B}^R(y) - \mathfrak{B}(y)]\,\mathrm{d}y \\ &\quad + \int_{|y| > M+r} G(x-y)[\mathfrak{B}^R(y) - \mathfrak{B}(y)]\,\mathrm{d}y \\ &=: I_1(x) + I_2(x).\end{aligned}$$

Now, by Young's inequality,

$$\|I_1\|_{L^{2,\infty}} \leq \|G\|_{L^{2,\infty}}\|\mathfrak{B}^R - \mathfrak{B}\|_{L^1(B_{M+r})}.$$

If $|x| \le r$, then

$$\begin{aligned} |I_2(x)| &\le \left(\int_{|z|\ge M} |G(z)|^4 \right)^{1/4} \|\mathfrak{B}^R - \mathfrak{B}\|_{L^{4/3}(\mathbb{R}^2)} \\ &\le cM^{-1/2} \|\mathfrak{B}^R - \mathfrak{B}\|_{L^{4/3}(\mathbb{R}^2)}, \end{aligned}$$

since $|G(x)| \le c/|x|$. Hence

$$\|\boldsymbol{u}^R - \boldsymbol{u}\|_{L^{2,\infty}(B_r)} \le \|G\|_{L^{2,\infty}} \|\mathfrak{B}^R - \mathfrak{B}\|_{L^1(B_{M+r})} + crM^{-1/2} \|\mathfrak{B}^R - \mathfrak{B}\|_{L^{4/3}(\mathbb{R}^2)}$$

Since $\mathfrak{B}^R - \mathfrak{B}$ is bounded in $L^4(0, T; L^{4/3}(\mathbb{R}^2))$,

$$\int_0^T \|\boldsymbol{u}^R - \boldsymbol{u}\|^2_{L^{2,\infty}(B_r)} \,\mathrm{d}t \le c \int_0^T \|\mathfrak{B}^R - \mathfrak{B}\|^2_{L^1(B_{M+r})} \,\mathrm{d}t + crM^{-1/2}.$$

Thus, given an arbitrary $\delta > 0$ we first pick M sufficiently large so that $crM^{-1/2} < \delta/2$, and then choose R sufficiently large to make the first term at most $\delta/2$. This completes the proof. □

This local strong convergence allows us to pass to the limit in the nonlinear terms: an argument similar to Proposition 4.7 will show that (after passing to a subsequence)

$$\mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R] \overset{*}{\rightharpoonup} (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}, \qquad \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] \overset{*}{\rightharpoonup} (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

in $L^2(0, T; V^*(\mathbb{R}^2))$ (see §2.2.4 of Chemin et al. (2006) for full details). Thus $(\boldsymbol{u}, \boldsymbol{B})$ is indeed a weak solution of (1.2), which completes the proof of Theorem 4.3 in the case $\Omega = \mathbb{R}^2$.

4.2.3 Uniqueness

We now prove that weak solutions are unique. Note that the following proof applies equally in all three cases of Theorem 4.1.

Proposition 4.9. *Let* $(\boldsymbol{u}_j, \boldsymbol{B}_j)$, $j = 1, 2$, *be two weak solutions with the same initial condition* $\boldsymbol{B}_j(0) = \boldsymbol{B}_0$, *such that*

$$\begin{aligned} \boldsymbol{u}_j &\in L^\infty(0, T; L^{2,\infty}(\Omega)) \cap L^2(0, T; V(\Omega)), \\ \boldsymbol{B}_j &\in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; V(\Omega)), \\ \frac{\partial \boldsymbol{B}_j}{\partial t} &\in L^2(0, T; V^*(\Omega)). \end{aligned}$$

Then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\boldsymbol{B}_1 = \boldsymbol{B}_2$ as functions in the above spaces.

Proof. Take the equations for $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2)$ and subtract: writing $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\boldsymbol{z} = \boldsymbol{B}_1 - \boldsymbol{B}_2$, we obtain

$$0 = \langle \nu \nabla \boldsymbol{w}, \nabla \boldsymbol{v} \rangle + \langle (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{v}, \boldsymbol{z} \rangle - \langle (\boldsymbol{z} \cdot \nabla)\boldsymbol{v}, \boldsymbol{B}_2 \rangle, \tag{4.6a}$$

$$\begin{aligned} \left\langle \frac{\partial \boldsymbol{z}}{\partial t}, \boldsymbol{v} \right\rangle &= \langle \eta \nabla \boldsymbol{z}, \nabla \boldsymbol{v} \rangle + \langle (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{v}, \boldsymbol{w} \rangle - \langle (\boldsymbol{z} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u}_2 \rangle \\ &\quad - \langle (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{v}, \boldsymbol{z} \rangle + \langle (\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{B}_2 \rangle. \end{aligned} \tag{4.6b}$$

Let $\boldsymbol{v} = \boldsymbol{w}(t)$ in (4.6a) and use the weak Ladyzhenskaya inequality (3.4) to get

$$\nu \|\nabla \boldsymbol{w}\|_{L^2} \le c \|\boldsymbol{z}\|_{L^2}^{1/2} \|\nabla \boldsymbol{z}\|_{L^2}^{1/2} (\|\nabla \boldsymbol{B}_1\|_{L^2}^{1/2} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{1/2}), \tag{4.7}$$

since $\boldsymbol{B}_j \in L^\infty(0, T; L^2(\Omega))$. By the elliptic regularity arguments from Section 4.1, we obtain

$$\|\boldsymbol{w}\|_{L^{2,\infty}} \le c \|\boldsymbol{z}\|_{L^2}. \tag{4.8}$$

Using the weak Ladyzhenskaya inequality (3.4), we obtain bounds in L^4 as follows:

$$\|\boldsymbol{w}\|_{L^4} \le \frac{c}{\nu} \|\boldsymbol{z}\|_{L^2}^{3/4} \|\nabla \boldsymbol{z}\|_{L^2}^{1/4} (\|\nabla \boldsymbol{B}_1\|_{L^2}^{1/4} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{1/4}) \tag{4.9}$$

(using elliptic regularity arguments from Section 4.1). By a similar argument (taking the inner product of (1.2a) with $\boldsymbol{u}_j$) we obtain

$$\|\boldsymbol{u}_j\|_{L^4} \le c \|\nabla \boldsymbol{B}_j\|_{L^2}^{1/2}. \tag{4.10}$$

As $\boldsymbol{z} \in L^2(0, T; V(\Omega))$ and $\frac{\partial \boldsymbol{z}}{\partial t} \in L^2(0, T; V^*(\Omega))$, we can take $\boldsymbol{v} = \boldsymbol{z}(t)$ in (4.6b) to obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{z}\|_{L^2}^2 + \eta \|\nabla \boldsymbol{z}\|_{L^2}^2 \\ &\le c \|\nabla \boldsymbol{z}\|_{L^2} \|\boldsymbol{w}\|_{L^4} (\|\boldsymbol{B}_1\|_{L^4} + \|\boldsymbol{B}_2\|_{L^4}) + c \|\nabla \boldsymbol{z}\|_{L^2} \|\boldsymbol{z}\|_{L^4} (\|\boldsymbol{u}_1\|_{L^4} + \|\boldsymbol{u}_2\|_{L^4}) \\ &\le c \underbrace{\|\nabla \boldsymbol{z}\|_{L^2} \|\boldsymbol{w}\|_{L^4} (\|\boldsymbol{B}_1\|_{L^4} + \|\boldsymbol{B}_2\|_{L^4})}_{\mathrm{I}} + c \underbrace{\|\nabla \boldsymbol{z}\|_{L^2} \|\boldsymbol{z}\|_{L^4} (\|\boldsymbol{u}_1\|_{L^4} + \|\boldsymbol{u}_2\|_{L^4})}_{\mathrm{II}}. \end{aligned}$$

By using (4.9) and Young's inequality with $\varepsilon = \frac{\eta}{4}$ for $8/5$ and $8/3$, we obtain

$$\begin{aligned}
\mathrm{I} &\le \frac{c}{\nu}\|\boldsymbol{z}\|_{L^2}^{3/4}\|\nabla \boldsymbol{z}\|_{L^2}^{5/4}(\|\nabla \boldsymbol{B}_1\|_{L^2}^{1/4} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{1/4})(\|\nabla \boldsymbol{B}_1\|_{L^2}^{1/2} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{1/2}) \\
&\le \frac{c}{\nu}\|\boldsymbol{z}\|_{L^2}^{3/4}\|\nabla \boldsymbol{z}\|_{L^2}^{5/4}(\|\nabla \boldsymbol{B}_1\|_{L^2}^{3/4} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{3/4}) \\
&\le \frac{\eta}{4}\|\nabla \boldsymbol{z}\|_{L^2}^2 + \frac{c}{\nu\eta^{5/3}}\|\boldsymbol{z}\|_{L^2}^2(\|\nabla \boldsymbol{B}_1\|_{L^2}^2 + \|\nabla \boldsymbol{B}_2\|_{L^2}^2).
\end{aligned}$$

Similarly, by using (4.10) and Young's inequality with $\varepsilon = \frac{\eta}{4}$ for $4/3$ and 4, we obtain

$$\begin{aligned}
\mathrm{II} &\le \|\boldsymbol{z}\|_{L^2}^{1/2}\|\nabla \boldsymbol{z}\|_{L^2}^{3/2}(\|\nabla \boldsymbol{B}_1\|_{L^2}^{1/2} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{1/2}) \\
&\le \frac{\eta}{4}\|\nabla \boldsymbol{z}\|_{L^2}^2 + \frac{c}{\eta^3}\|\boldsymbol{z}\|_{L^2}^2(\|\nabla \boldsymbol{B}_1\|_{L^2}^2 + \|\nabla \boldsymbol{B}_2\|_{L^2}^2).
\end{aligned}$$

Hence using (4.9) and Young's inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{z}\|_{L^2}^2 + \eta\|\nabla \boldsymbol{z}\|_{L^2}^2 \le c(\nu,\eta)\|\boldsymbol{z}\|_{L^2}^2(\|\nabla \boldsymbol{B}_1\|_{L^2}^2 + \|\nabla \boldsymbol{B}_2\|_{L^2}^2). \tag{4.11}$$

As $\boldsymbol{B}_j \in L^2(0,T;V(\Omega))$, and $\boldsymbol{z}_0 = 0$, Gronwall's inequality shows that $\|\boldsymbol{z}\|_{L^2} = \|\nabla \boldsymbol{z}\|_{L^2} = 0$ on $[0,T]$. This implies $\|\boldsymbol{w}\|_{L^{2,\infty}} = \|\nabla \boldsymbol{w}\|_{L^2} = 0$ on $[0,T]$, and uniqueness of weak solutions follows. □

This completes the proof of Theorem 4.3.

4.3 Higher-Order Regularity Estimates

In this section, we prove the second part of Theorem 4.1; that is, that the solution $(\boldsymbol{u}, \boldsymbol{B})$ becomes smooth after an arbitrarily short time $\varepsilon > 0$. In particular, we prove that if we start with initial data in $H^k(\Omega)$, then the solution stays in $H^k(\Omega)$ for all time.

Theorem 4.10. *Let $k \in \mathbb{N}$. Suppose $\boldsymbol{B}_0 \in H^k(\Omega)$ with $\nabla \cdot \boldsymbol{B}_0 = 0$. Then, for any $T > 0$, the unique weak solution of* (1.2) *satisfies*

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0,T;H^k(\Omega)) \cap L^2(0,T;H^{k+1}(\Omega)).$$

Proof. We use induction on k; we show only the formal estimates (which can be made rigorous using the same methods as in the last section). First, suppose $\boldsymbol{B}_0 \in H^1(\Omega)$ with $\nabla \cdot \boldsymbol{B}_0 = 0$. Take the inner product of (1.2a) with $-\Delta \boldsymbol{u}$, the inner product of

(1.2b) with $-\Delta \boldsymbol{B}$, and add:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{B}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 + \eta\|\Delta \boldsymbol{B}\|_{L^2}^2$$
$$= \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, \Delta \boldsymbol{B}\rangle - \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \Delta \boldsymbol{B}\rangle - \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \Delta \boldsymbol{u}\rangle.$$

Using Young's inequality we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{B}\|_{L^2}^2 + \frac{\nu}{2}\|\Delta \boldsymbol{u}\|_{L^2}^2 + \frac{\eta}{2}\|\Delta \boldsymbol{B}\|_{L^2}^2 \\ &\le c\|\nabla \boldsymbol{B}\|_{L^2}^2\left(\|\boldsymbol{u}\|_{L^{2,\infty}}^2\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{u}\|_{L^2}^2\|\boldsymbol{B}\|_{L^2}^2 + \|\boldsymbol{B}\|_{L^2}^2\|\nabla \boldsymbol{B}\|_{L^2}^2\right).\end{aligned} \tag{4.12}$$

Since the integral of the last bracket is finite, by Gronwall's inequality we get that $\boldsymbol{B} \in L^\infty(0,T;H^1(\Omega))$. Hence, by (4.12), $\boldsymbol{u}, \boldsymbol{B} \in L^2(0,T;H^2(\Omega))$. Finally, take the inner product of (1.2a) with $\boldsymbol{u}$ to obtain

$$\nu\|\nabla \boldsymbol{u}\|_{L^2}^2 \le \|\boldsymbol{B}\|_{L^4}^2\|\nabla \boldsymbol{u}\|_{L^2},$$

so

$$\nu\|\nabla \boldsymbol{u}\|_{L^2} \le \|\boldsymbol{B}\|_{L^4}^2 \le c\|\boldsymbol{B}\|_{L^2}\|\nabla \boldsymbol{B}\|_{L^2},$$

and since the right-hand side is bounded, $\boldsymbol{u} \in L^\infty(0,T;H^1(\Omega))$.

For the induction step, let $k \ge 2$, and let $\boldsymbol{B}_0 \in H^k(\Omega)$ with $\nabla\cdot\boldsymbol{B}_0 = 0$. Suppose

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0,T;H^{k-1}(\Omega)) \cap L^2(0,T;H^k(\Omega)).$$

Take the inner product of (1.2a) with $(-1)^k\Delta^k\boldsymbol{u}$, the inner product of (1.2b) with $(-1)^k\Delta^k\boldsymbol{B}$, and add:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_{H^k}^2 + \nu\|\boldsymbol{u}\|_{H^{k+1}}^2 + \eta\|\boldsymbol{B}\|_{H^{k+1}}^2 \\ &= (-1)^k\left[\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \Delta^k\boldsymbol{B}\rangle + \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \Delta^k\boldsymbol{u}\rangle - \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, \Delta^k\boldsymbol{B}\rangle\right] \\ &\le c\left[\|(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\|_{H^k}\|\boldsymbol{B}\|_{H^k} + \|(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\|_{H^k}\|\boldsymbol{u}\|_{H^k} + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\|_{H^k}\|\boldsymbol{B}\|_{H^k}\right] \\ &\le c\left[\|\boldsymbol{B}\|_{H^k}^2\|\boldsymbol{u}\|_{H^{k+1}} + 2\|\boldsymbol{B}\|_{H^k}\|\boldsymbol{B}\|_{H^{k+1}}\|\boldsymbol{u}\|_{H^k}\right] \\ &\le \frac{\nu}{2}\|\boldsymbol{u}\|_{H^{k+1}}^2 + \frac{\eta}{2}\|\boldsymbol{B}\|_{H^{k+1}}^2 + c\left(\|\boldsymbol{u}\|_{H^k}^2 + \|\boldsymbol{B}\|_{H^k}^2\right)\|\boldsymbol{B}\|_{H^k}^2,\end{aligned}$$

where we have used the fact that H^k is an algebra for $k \ge 2$. We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_{H^k}^2 + \nu\|\boldsymbol{u}\|_{H^{k+1}}^2 + \eta\|\boldsymbol{B}\|_{H^{k+1}}^2 \le c\|\boldsymbol{B}\|_{H^k}^2\left(\|\boldsymbol{u}\|_{H^k}^2 + \|\boldsymbol{B}\|_{H^k}^2\right). \tag{4.13}$$

Since the integral of the last bracket is finite, by Gronwall's inequality we get that $\boldsymbol{B} \in L^\infty(0,T;H^k(\Omega))$, and hence reusing this bound in (4.13) yields $\boldsymbol{u}, \boldsymbol{B} \in L^2(0,T;H^{k+1}(\Omega))$.

If $k \geq 3$ take the inner product of (1.2a) with $(-1)^{k-1}\Delta^{k-1}\boldsymbol{u}$ to obtain

$$\nu\|\boldsymbol{u}\|_{H^k}^2 \leq c\|(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\|_{H^{k-1}}\|\boldsymbol{u}\|_{H^{k-1}} \leq \|\boldsymbol{B}\|_{H^{k-1}}\|\boldsymbol{B}\|_{H^k}\|\boldsymbol{u}\|_{H^{k-1}},$$

since H^{k-1} is an algebra when $k \geq 3$. Since the right-hand side is bounded, $\boldsymbol{u} \in L^\infty(0,T;H^k(\Omega))$. In the case $k = 2$, since H^1 is not an algebra, we must instead take the inner product of (1.2a) with $-\Delta\boldsymbol{u}$ and estimate as follows:

$$\nu\|\Delta\boldsymbol{u}\|_{L^2}^2 \leq |\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \Delta\boldsymbol{u}\rangle| \leq \|\boldsymbol{B}\|_{L^4}\|\nabla\boldsymbol{B}\|_{L^4}\|\Delta\boldsymbol{u}\|_{L^2},$$

so

$$\|\Delta\boldsymbol{u}\|_{L^2} \leq \|\boldsymbol{B}\|_{L^2}^{1/2}\|\nabla\boldsymbol{B}\|_{L^2}\|\Delta\boldsymbol{B}\|_{L^2}^{1/2},$$

and since the right-hand side is bounded, $\boldsymbol{u} \in L^\infty(0,T;H^2(\Omega))$. □

An immediate corollary of Theorem 4.10 is that the solution $(\boldsymbol{u}, \boldsymbol{B})$ becomes smooth after an arbitrarily short time $\varepsilon > 0$, which completes the proof of Theorem 4.1.

Corollary 4.11. *Given $T > \varepsilon > 0$ and $k \in \mathbb{N}$, the unique weak solution of* (1.2) *satisfies $\boldsymbol{u}, \boldsymbol{B} \in L^\infty(\varepsilon, T; H^k(\Omega))$.*

Proof. Fix $\varepsilon > 0$. We already know that $\boldsymbol{u}, \boldsymbol{B} \in L^2(0,T;H^1(\Omega))$, so for some time $t_1 < \varepsilon/2$, $\boldsymbol{u}(t_1), \boldsymbol{B}(t_1) \in H^1(\Omega)$. Applying Theorem 4.10, we obtain

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(\varepsilon/2, T; H^1(\Omega)) \cap L^2(\varepsilon/2, T; H^2(\Omega)).$$

Furthermore, if we know that

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(\varepsilon(1-2^{1-k}), T; H^{k-1}(\Omega)) \cap L^2(\varepsilon(1-2^{1-k}), T; H^k(\Omega)),$$

then there is some time t_k such that $\varepsilon(1-2^{1-k}) < t_k < \varepsilon(1-2^{-k})$ and $\boldsymbol{u}(t_k), \boldsymbol{B}(t_k) \in H^k(\Omega)$, and so applying Theorem 4.10, we obtain

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(\varepsilon(1-2^{-k}), T; H^k(\Omega)) \cap L^2(\varepsilon(1-2^{-k}), T; H^{k+1}(\Omega)).$$

The result follows by induction on k. □

4.4 The 3D Case

It is straightforward to adapt the methods of Section 4.2 to the 3D case to prove global existence — but *not* uniqueness — of at least one weak solution to (1.2) in 3D. Indeed, for $\Omega \subset \mathbb{R}^3$ in the analogue of Theorem 4.1, given an initial condition $\boldsymbol{B}_0 \in H(\Omega)$ there exists at least one weak solution $(\boldsymbol{u}, \boldsymbol{B})$ of (1.2) on $(0,T)$ with

$$\begin{aligned}
\boldsymbol{u} &\in L^\infty(0,T;L^{3/2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\boldsymbol{B} &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\frac{\partial \boldsymbol{B}}{\partial t} &\in L^{24/19}(0,T;V^*(\Omega)),
\end{aligned}$$

satisfying the initial data in the sense that $\lim_{t\to 0^+} \boldsymbol{B}(t) = \boldsymbol{B}_0$ as a limit in $V^*(\Omega)$.

The key differences come from the elliptic regularity and the consequent interpolation inequalities. In 3D, the solution of the Stokes equation (4.1) satisfies $\boldsymbol{u} \in L^{3/2,\infty}(\Omega)$ whenever $\boldsymbol{f} \in L^1(\Omega)$. The standard 3D Ladyzhenskaya inequality

$$\|f\|_{L^4} \le c\|f\|_{L^2}^{1/4}\|f\|_{H^1}^{3/4} \tag{4.14}$$

is then sufficient; using that and the interpolation inequality

$$\|f\|_{L^4} \le c\|f\|_{L^{3/2,\infty}}^{1/6}\|f\|_{L^6}^{5/6} \le c\|f\|_{L^{3/2,\infty}}^{1/6}\|f\|_{H^1}^{5/6} \tag{4.15}$$

it is straightforward to show the corresponding energy estimates to Proposition 4.4 in the case when Ω is a bounded domain in $\mathbb{R}^3$:

$$\begin{aligned}
\boldsymbol{u}^m &\text{ is uniformly bounded in } L^\infty(0,T;L^{3/2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\boldsymbol{B}^m &\text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\frac{\partial \boldsymbol{B}^m}{\partial t} &\text{ is uniformly bounded in } L^{24/19}(0,T;V^*(\Omega)).
\end{aligned}$$

The Aubin-Lions compactness lemma then shows that $\boldsymbol{B}^m \to \boldsymbol{B}$ strongly in both $L^2(0,T;L^2(\Omega))$ and $C^0([0,T];V^*(\Omega))$ (thus the initial data is attained in this sense). It is easy to adjust Proposition 4.6 to show $\boldsymbol{u}^m \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{3/2,\infty}(\Omega))$, using (4.14) and (4.15). Modifying Proposition 4.7 yields convergence of the nonlinear terms in $L^{24/19}(0,T;V^*(\Omega))$, and hence $(\boldsymbol{u}, \boldsymbol{B})$ is a weak solution of (1.2).

It is also routine to modify the method of Section 4.2.2 to prove existence in the case $\Omega = \mathbb{R}^3$. One modifies Proposition 4.8 to show that $\boldsymbol{u}^R \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{3/2,\infty}(K))$ for any compact subset $K \subset \mathbb{R}^3$, by using the embedding $\dot{W}^{1,1}(\mathbb{R}^3) \subset L^{3/2}(\mathbb{R}^3)$, and $|G| = |\partial_k \boldsymbol{U}| \le c/|x|^2 \in L^{3/2,\infty}(\mathbb{R}^3)$.

Chapter 5

Commutator Estimates

In this chapter we prove a new commutator estimate for fractional derivatives that will prove vital as an intermediate step to proving local existence of solutions to the non-resistive MHD equations (1.5) and the non-resistive Stokes-MHD system (1.6) in Chapter 6.

The main difficulty in proving local existence for equations (1.5) with diffusion only in the $\boldsymbol{u}$ equation (i.e. $\eta = 0$) stems from the nonlinear terms. Naively, H^s is an algebra for $s > n/2$, so one obtains

$$|\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}\rangle_{H^s}| \le \|\boldsymbol{u}\|_{H^s}\|\nabla\boldsymbol{v}\|_{H^s}\|\boldsymbol{w}\|_{H^s}.$$

For three of the four nonlinear terms, this is sufficient, but for the $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ term we must estimate $\|\nabla\boldsymbol{B}\|_{H^s}$, and if we start with $\boldsymbol{B}_0 \in H^s$ we have no control over the H^s norm of $\nabla\boldsymbol{B}$ because there is no smoothing for $\boldsymbol{B}$. We will show that for $s > n/2$ one can in fact obtain the bound

$$|\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{B},\boldsymbol{B}\rangle_{H^s}| \le c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}^2.$$

This is a consequence of a new commutator estimate applicable to the nonlinear terms, which we prove in this chapter.

To describe this, recall from (2.13) that J^s and Λ^s denote fractional derivative operators defined in terms of Fourier transforms as follows:

$$\mathscr{F}[J^s f](\xi) = (1+|\xi|^2)^{s/2}\hat{f}(\xi), \qquad \mathscr{F}[\Lambda^s f](\xi) = |\xi|^s\hat{f}(\xi).$$

It was proved in Kato & Ponce (1988) that, for $s \ge 0$ and $1 < p < \infty$, the nonlinear terms satisfy the following estimate:

$$\|J^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}]-(\boldsymbol{u}\cdot\nabla)(J^s\boldsymbol{B})\|_{L^p}\le c(\|\nabla\boldsymbol{u}\|_{L^\infty}\|J^{s-1}\nabla\boldsymbol{B}\|_{L^p}+\|J^s\boldsymbol{u}\|_{L^p}\|\nabla\boldsymbol{B}\|_{L^\infty})$$

which, for $p=2$ and $s>n/2$, implies the following:

$$\|J^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}]-(\boldsymbol{u}\cdot\nabla)(J^s\boldsymbol{B})\|_{L^2}\le c(\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}+\|\boldsymbol{u}\|_{H^s}\|\nabla\boldsymbol{B}\|_{H^s}). \tag{5.1}$$

Once again, however, estimate (5.1) cannot immediately be applied to our system of equations, because the second term on the right-hand side still contains $\|\nabla\boldsymbol{B}\|_{H^s}$.

In this chapter we prove a new commutator estimate, involving Λ instead of J: in Section 5.1 we prove that

$$\|\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}]-(\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B})\|_{L^2}\le c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s} \tag{5.2}$$

for any $s>n/2$. Note that (5.2) only contains the first of the two terms on the right-hand side of (5.1), which will enable us to prove local existence of solutions to both (1.5) and (1.6) in the case $\eta=0$.

In the case $s=n/2$, one can prove (see Appendix A) that

$$\|\Lambda^{n/2}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}]-(\boldsymbol{u}\cdot\nabla)(\Lambda^{n/2}\boldsymbol{B})\|_{L^2}\le c(\|\nabla\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\boldsymbol{B}\|_{\dot{H}^{n/2}}+\|\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\nabla\boldsymbol{B}\|_{\dot{H}^{n/2}}).$$

Unfortunately it is impossible in this case to get rid of the second term on the right-hand side: in Section 5.2, we exhibit a counterexample to show that (5.2) does not hold in the case $s=n/2$, at least for $n=2$, even if $\boldsymbol{u}$ and $\boldsymbol{B}$ are required to be divergence-free; this therefore suggests that proving local existence in $H^{n/2}$ (if possible) would require a more refined technique.

5.1 Commutator Estimate in $H^s(\mathbb{R}^n)$ for $s>n/2$

In this section, we prove the following commutator estimate.

Theorem 5.1. *Given $s>n/2$, there is a constant $c=c(n,s)$ such that, for all $\boldsymbol{u},\boldsymbol{B}$ with $\nabla\boldsymbol{u},\boldsymbol{B}\in H^s(\mathbb{R}^n)$,*

$$\|[\Lambda^s,(\boldsymbol{u}\cdot\nabla)]\boldsymbol{B}\|_{L^2}:=\|\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}]-(\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B})\|_{L^2}\le c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}. \tag{5.3}$$

Before embarking on the proof, we note that a priori the left-hand side makes sense only when $\boldsymbol{u},\nabla\boldsymbol{B}\in H^s(\mathbb{R}^n)$; however, the right-hand side is finite when

$\nabla \boldsymbol{u}, \boldsymbol{B} \in H^s(\mathbb{R}^n)$, and since both sides are linear in $\boldsymbol{u}$ and $\boldsymbol{B}$ it suffices to prove the inequality for $\boldsymbol{u}, \boldsymbol{B} \in C_c^\infty(\mathbb{R}^n)$ and use the density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.

Proof. Let $\boldsymbol{u}, \boldsymbol{B} \in C_c^\infty(\mathbb{R}^n)$. First, note that

$$\mathscr{F}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}_k](\xi) = \sum_{j=1}^n (\widehat{\boldsymbol{u}_j \partial_j \boldsymbol{B}_k})(\xi) = \sum_{j=1}^n \int \hat{\boldsymbol{u}}_j(\zeta)(\xi - \zeta)_j \hat{\boldsymbol{B}}_k(\xi - \zeta)\,\mathrm{d}\zeta,$$

so

$$\mathscr{F}[\Lambda^s[(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}_k]](\xi) = |\xi|^s \sum_{j=1}^n \int \hat{\boldsymbol{u}}_j(\zeta)(\xi - \zeta)_j \hat{\boldsymbol{B}}_k(\xi - \zeta)\,\mathrm{d}\zeta.$$

Similarly,

$$\mathscr{F}[(\boldsymbol{u} \cdot \nabla)(\Lambda^s \boldsymbol{B}_k)](\xi) = \sum_{j=1}^n \int \hat{\boldsymbol{u}}_j(\zeta)(\xi - \zeta)_j |\xi - \zeta|^s \hat{\boldsymbol{B}}_k(\xi - \zeta)\,\mathrm{d}\zeta.$$

Therefore the Fourier transform of $\Lambda^s[(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}] - (\boldsymbol{u} \cdot \nabla)(\Lambda^s \boldsymbol{B})$ is

$$\sum_{j=1}^n \int (|\xi|^s - |\xi - \zeta|^s)\hat{\boldsymbol{u}}_j(\zeta)(\xi - \zeta)_j \hat{\boldsymbol{B}}_k(\xi - \zeta)\,\mathrm{d}\zeta;$$

by Parseval's identity it suffices to bound this in L^2.

We split the integral into the two regions $|\zeta| < |\xi|/2$ and $|\zeta| \geq |\xi|/2$. In the first region $|\zeta| < |\xi|/2$, we use the inequality

$$||\xi|^s - |\xi - \zeta|^s| \leq c|\xi - \zeta|^{s-1}|\zeta| \tag{5.4}$$

(whose proof we postpone to Lemma 5.2) to obtain

$$\sum_{j=1}^n \int_{|\zeta|<|\xi|/2} (|\xi|^s - |\xi - \zeta|^s)\hat{\boldsymbol{u}}_j(\zeta)(\xi - \zeta)_j \hat{\boldsymbol{B}}_k(\xi - \zeta)\,\mathrm{d}\zeta$$
$$\leq c \int |\zeta||\hat{\boldsymbol{u}}(\zeta)||\xi - \zeta|^s|\hat{\boldsymbol{B}}(\xi - \zeta)|\,\mathrm{d}\zeta.$$

By Young's inequality, the L^2 norm of the above integral expression is bounded above by

$$\begin{aligned}
&\left\| |\zeta||\hat{\boldsymbol{u}}(\zeta)| \right\|_{L^1} \left\| |\eta|^s |\hat{\boldsymbol{B}}(\eta)| \right\|_{L^2} \\
&\qquad \le \left\| \frac{1}{(1+|\zeta|^2)^{s/2}} \right\|_{L^2} \left\| (1+|\zeta|^2)^{s/2} |\zeta||\hat{\boldsymbol{u}}(\zeta)| \right\|_{L^2} \left\| |\eta|^s |\hat{\boldsymbol{B}}(\eta)| \right\|_{L^2} \\
&\qquad \le c\|\nabla \boldsymbol{u}\|_{H^s} \|\boldsymbol{B}\|_{\dot{H}^s},
\end{aligned}$$

since $(1+|\zeta|^2)^{-s/2} \in L^2$ as $s > n/2$.

In the second region $|\zeta| \ge |\xi|/2$, we have $|\xi| \le 2|\zeta|$ and $|\xi - \zeta| \le 3|\zeta|$. So

$$||\xi|^s - |\xi - \zeta|^s| \le c|\zeta|^s,$$

hence

$$\begin{aligned}
&\sum_{j=1}^n \int_{|\zeta|\ge|\xi|/2} (|\xi|^s - |\xi-\zeta|^s) \hat{\boldsymbol{u}}_j(\zeta)(\xi-\zeta)_j \hat{\boldsymbol{B}}_k(\xi-\zeta)\,\mathrm{d}\zeta \\
&\qquad\qquad \le c \int |\zeta|^{s+1} |\hat{\boldsymbol{u}}(\zeta)||\hat{\boldsymbol{B}}(\xi-\zeta)|\,\mathrm{d}\zeta.
\end{aligned}$$

The L^2 norm of the above integral expression is bounded by

$$\begin{aligned}
&\left\| |\zeta|^{s+1} |\hat{\boldsymbol{u}}(\zeta)| \right\|_{L^2} \left\| |\hat{\boldsymbol{B}}(\eta)| \right\|_{L^1} \\
&\qquad \le \left\| |\zeta|^{s+1} |\hat{\boldsymbol{u}}(\zeta)| \right\|_{L^2} \left\| \frac{1}{(1+|\eta|^2)^{s/2}} \right\|_{L^2} \left\| (1+|\eta|^2)^{s/2} |\hat{\boldsymbol{B}}(\eta)| \right\|_{L^2} \\
&\qquad \le c\|\nabla \boldsymbol{u}\|_{\dot{H}^s} \|\boldsymbol{B}\|_{H^s},
\end{aligned}$$

since $s > n/2$. This completes the proof when $\boldsymbol{u}, \boldsymbol{B} \in C_c^\infty(\mathbb{R}^n)$, and the general case follows by density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$. □

It remains to prove the inequality (5.4).

Lemma 5.2. *If $s \ge 1$ and $|b| < |a|/2$, then*

$$||a|^s - |a-b|^s| \le c|a-b|^{s-1}|b|.$$

Proof. Given a and b, let $h(t) = |a - tb|^s$. As $|b| < |a|/2$, h is smooth on $[0,1]$. Now

$$h'(t) = -s|a-tb|^{s-2}(a-tb)\cdot b,$$

so applying the Mean Value Theorem to h on $[0,1]$ we obtain

$$||a|^s - |a-b|^s| \le \max_{t\in[0,1]} |h'(t)| \le s|b| \max_{t\in[0,1]} |a-tb|^{s-1}.$$

As $|b| < |a|/2$, for all $t \in [0,1]$,

$$\frac{|a|}{2} \le |a-tb| \le \frac{3|a|}{2};$$

in particular $\frac{|a|}{2} \le |a-b|$ and so

$$|a-tb| \le \frac{3|a|}{2} \le 3|a-b|,$$

and since $s \ge 1$ we have

$$||a|^s - |a-b|^s| \le 3^{s-1}s|b||a-b|^{s-1}. \qquad \square$$

Using the fact that, when $\boldsymbol{u}$ is divergence-free,

$$\langle(\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B}), \Lambda^s\boldsymbol{B}\rangle = 0,$$

we immediately obtain the following corollary of Theorem 5.1.

Corollary 5.3. *Given $s > n/2$, there is a constant $c = c(n,s)$ such that, for all $\boldsymbol{u}, \boldsymbol{B}$ with $\nabla\boldsymbol{u}, \boldsymbol{B} \in H^s(\mathbb{R}^n)$ and $\nabla\cdot\boldsymbol{u} = 0$,*

$$|\langle\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}], \Lambda^s\boldsymbol{B}\rangle| \le c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}^2.$$

5.2 A Counterexample to Theorem 5.1 in $H^1(\mathbb{R}^2)$

In this section, we show that the result of Theorem 5.1 cannot be extended to the case $s = 1$ when $n = 2$: we give an example to show that the inequality

$$\|\partial_k[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] - (\boldsymbol{u}\cdot\nabla)(\partial_k\boldsymbol{B})\|_{L^2} \le c\|\nabla\boldsymbol{u}\|_{H^1}\|\boldsymbol{B}\|_{H^1} \tag{5.5}$$

cannot hold in dimension 2, by exhibiting a pair of divergence-free functions $\boldsymbol{u}$ and $\boldsymbol{B}$ for which the right-hand side is finite, but the left-hand side is infinite. As a result, since Theorem 5.1 is key to proving local existence with initial data in H^s for $s > n/2$, if we were to try to prove local existence with initial data in $H^{n/2}$ then a different approach would be required.

Since we have one full derivative, we can make an important simplification by means of the product rule: the inequality reduces to

$$\|((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}\|_{L^2} \leq c\|\nabla \boldsymbol{u}\|_{H^1}\|\boldsymbol{B}\|_{H^1}. \tag{5.6}$$

Now, Theorem 5.1 does not require $\boldsymbol{u}$ and $\boldsymbol{B}$ to be divergence-free, and it is possible to give an easier counterexample to (5.6) if we do not insist on the divergence-free requirement. However, in order to eliminate the possibility that (5.6) might hold for divergence-free vector fields, even if it does not hold in general, we present here a counterexample in which $\boldsymbol{u}$ and $\boldsymbol{B}$ are divergence-free.

Since we are in two dimensions, we may represent our divergence-free vector fields as $\boldsymbol{u} = \nabla^\perp \phi$ and $\boldsymbol{B} = \nabla^\perp \psi$ for some scalar functions ϕ and ψ; in other words,

$$\boldsymbol{u} = (\partial_2 \phi, -\partial_1 \phi), \qquad \boldsymbol{B} = (\partial_2 \psi, -\partial_1 \psi).$$

Thus

$$((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1 = (\partial_k \boldsymbol{u}_1)(\partial_1 \boldsymbol{B}_1) + (\partial_k \boldsymbol{u}_2)(\partial_2 \boldsymbol{B}_1)$$

becomes

$$((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1 = (\partial_k \partial_2 \phi)(\partial_1 \partial_2 \psi) - (\partial_k \partial_1 \phi)(\partial_2^2 \psi)$$

(one can treat the second component similarly). Taking Fourier transforms of both sides yields

$$\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1](\xi) = 16\pi^4 \int \underbrace{\zeta_k (\xi - \zeta)_2 [\zeta^\perp \cdot (\xi - \zeta)]}_{(*)} \hat{\phi}(\zeta)\hat{\psi}(\xi - \zeta)\,\mathrm{d}\zeta. \tag{5.7}$$

By choosing the support of $\hat{\phi}$ and $\hat{\psi}$ to lie in certain small sectors, we may bound the expression $(*)$ below by the absolute values of the respective components; that is,

$$\zeta_k (\xi - \zeta)_2 [\zeta^\perp \cdot (\xi - \zeta)] \geq M_\delta |\zeta|^2 |\xi - \zeta|^2.$$

This is made precise in the following lemma. (The proof thereof is largely elementary, using the bound $\sin x \geq 1 - \frac{2}{\pi}|x - \frac{\pi}{2}|$ for $x \in (0, \pi)$, and we postpone the details.)

Lemma 5.4. *Fix $0 < \delta < \frac{1}{\sqrt{2}}$. Suppose that $\zeta, \eta \in \mathbb{R}^2$ satisfy $|\arg \zeta - \frac{\pi}{4}| < \delta$, $|\arg \eta - \frac{3\pi}{4}| < \delta$. Then*

$$\frac{\zeta_k}{|\zeta|} \frac{\eta_2}{|\eta|} \frac{[\zeta^\perp \cdot \eta]}{|\zeta||\eta|} \geq \left(\tfrac{\sqrt{2}}{2} - \delta\right)^2 \left(1 - \tfrac{4\delta}{\pi}\right) =: M_\delta > 0.$$

Consider now ϕ and ψ of the form

$$\hat{\phi}(\zeta) = \frac{1}{|\zeta|^2(1+|\zeta|^2)^{1/2}} g(|\zeta|) h_1(\arg\zeta), \tag{5.8a}$$

$$\hat{\psi}(\eta) = \frac{1}{|\eta|(1+|\eta|^2)^{1/2}} g(|\eta|) h_2(\arg\eta), \tag{5.8b}$$

where

$$g(r) = \begin{cases} \frac{1}{r(\log r)^\alpha} & \text{for } r > e \\ 0 & \text{otherwise,} \end{cases}$$

$\alpha > 0$ will be chosen later, and

$$h_1(\theta) = \begin{cases} 1 & \text{for } \theta \in [\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta] \\ 0 & \text{for } \theta \notin [\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta] \end{cases}$$

and

$$h_2(\theta) = \begin{cases} 1 & \text{for } \theta \in [\frac{3\pi}{4} - \delta, \frac{3\pi}{4} + \delta] \\ 0 & \text{for } \theta \notin [\frac{3\pi}{4} - \delta, \frac{3\pi}{4} + \delta]. \end{cases}$$

Notice that

$$\|\nabla \boldsymbol{u}\|_{H^1}^2 = \|\nabla(\nabla^\perp \phi)\|_{H^1}^2 = \|(1+|\zeta|^2)^{1/2}|\zeta|^2 \hat{\phi}(\zeta)\|_{L^2}^2 = \int |g(|\zeta|) h_1(\arg\zeta)|^2 \,\mathrm{d}\zeta,$$

$$\|\boldsymbol{B}\|_{H^1}^2 = \|(\nabla^\perp \psi)\|_{H^1}^2 = \|(1+|\eta|^2)^{1/2}|\eta| \hat{\psi}(\eta)\|_{L^2}^2 = \int |g(|\eta|) h_2(\arg\eta)|^2 \,\mathrm{d}\eta,$$

and hence

$$\|\nabla \boldsymbol{u}\|_{H^1}^2 = \|\boldsymbol{B}\|_{H^1}^2 = 2\delta \int_e^\infty \frac{1}{r(\log r)^{2\alpha}} \,\mathrm{d}r = \frac{2\delta}{1-2\alpha} (\log r)^{1-2\alpha} \Big|_e^\infty$$

which is finite iff $\alpha > 1/2$.

However, by choosing ξ and ζ carefully — which we do in full detail shortly — we may bound the expression (5.7) below by

$$\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla) \boldsymbol{B}_1](\xi) \geq c \int_\Omega \frac{1}{|\zeta|} g(|\zeta|) g(|\xi - \zeta|) \,\mathrm{d}\zeta$$

for some sector Ω in Fourier space. For small ζ, $g(|\xi-\zeta|) \approx g(|\xi|)$, so

$$\begin{aligned}\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1](\xi) &\gtrsim cg(|\xi|) \int_\Omega \frac{1}{|\zeta|} g(|\zeta|)\,\mathrm{d}\zeta \\ &\approx \frac{c}{|\xi|(\log|\xi|)^\alpha} \int_1^{|\xi|} \frac{1}{|r|(\log r)^\alpha}\,\mathrm{d}r \\ &= \frac{c}{|\xi|(\log|\xi|)^{2\alpha-1}},\end{aligned}$$

and the right-hand side is in L^2 if and only if $\alpha > 3/4$. Hence choosing $1/2 < \alpha < 3/4$ will yield our counterexample.

To make this fully rigorous, we carefully choose at which ξ we evaluate (5.7), to ensure that both ζ and $\xi - \zeta$ fall into the ranges required in Lemma 5.4, and thus find a lower bound for (5.7). This is the content of the following lemma.

Lemma 5.5. *Let*

$$\begin{aligned}\Xi &:= \{\xi \in \mathbb{R}^2 : \arg\xi \in [\tfrac{3\pi}{4} - \tfrac{\delta}{2}, \tfrac{3\pi}{4} + \tfrac{\delta}{2}]\}, \\ \Upsilon_\xi &:= \{\zeta \in \mathbb{R}^2 : |\zeta| < |\xi| \sin\tfrac{\delta}{2} \ \textit{and} \ \arg\zeta \in [\tfrac{\pi}{4} - \delta, \tfrac{\pi}{4} + \delta]\}.\end{aligned}$$

Then

$$\xi \in \Xi,\ \zeta \in \Upsilon_\xi \implies \arg(\xi - \zeta) \in [\tfrac{3\pi}{4} - \delta, \tfrac{3\pi}{4} + \delta].$$

The situation is illustrated in figure 5.1: the light shaded region is Ξ, while the two darker shaded regions are Υ_ξ and $\xi - \Upsilon_\xi$. We postpone the proof of the lemma to the end of the section.

We now restrict the sectors Ξ and Υ_ξ to particular radii: setting $K = \sin\frac{\delta}{2}$, we let

$$\begin{aligned}X &:= \{\xi \in \mathbb{R}^2 : |\xi| > e/K \text{ and } \arg\xi \in [\tfrac{3\pi}{4} - \tfrac{\delta}{2}, \tfrac{3\pi}{4} + \tfrac{\delta}{2}]\} \subset \Xi, \\ Z_\xi &:= \{\zeta \in \mathbb{R}^2 : e < |\zeta| < K|\xi| \text{ and } \arg\zeta \in [\tfrac{\pi}{4} - \delta, \tfrac{\pi}{4} + \delta]\} \subset \Upsilon_\xi.\end{aligned}$$

By staying away from the origin, we ensure that $|\xi|$ and $(1+|\xi|^2)^{1/2}$ are comparable: indeed, note that

$$\frac{|\xi|}{(1+|\xi|^2)^{1/2}} \geq \frac{1}{\sqrt{2}} \qquad \text{for } |\xi| \geq 1. \tag{5.9}$$

Hence, for $\xi \in X$, using Lemmas 5.4 and 5.5 and estimate (5.9), equation (5.7) reduces to

$$\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1](\xi) \geq c \int_{Z_\xi} \frac{1}{|\zeta|} g(|\zeta|) g(|\xi - \zeta|)\,\mathrm{d}\zeta$$

for $c = 8\pi^4 M_\delta$.

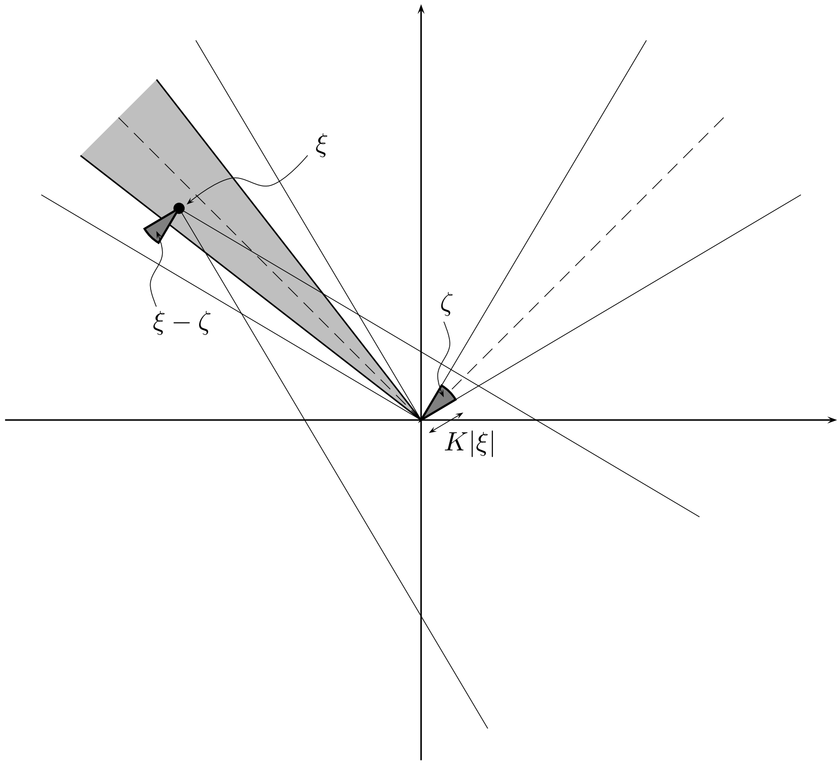

Figure 5.1: A plot showing the sectors (in Fourier space) in which we need ξ and ζ to lie, where $K = \sin\frac{\delta}{2}$.

When $|\zeta| < K|\xi|$, we get $(1-K)|\xi| < |\xi - \zeta| < (1+K)|\xi|$; and as $\delta < \pi/3$, $(1-K) > K$, ensuring that $g((1-K)|\xi|) > 0$. Thus for $\xi \in X$ and $\zeta \in Z_\xi$,

$$g(|\xi - \zeta|) \geq g((1+K)|\xi|) > 0.$$

Thus

$$\begin{aligned}
\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1](\xi) &\geq c g((1+K)|\xi|) \int_{Z_\xi} \frac{1}{|\zeta|} g(|\zeta|) \, \mathrm{d}\zeta \\
&= 2\delta c g((1+K)|\xi|) \int_e^{K|\xi|} \frac{1}{r(\log r)^\alpha} \, \mathrm{d}r.
\end{aligned}$$

Since

$$\int_e^{K|\xi|} \frac{1}{r(\log r)^\alpha} \, \mathrm{d}r = (\log r)^{1-\alpha} \Big|_e^{K|\xi|} = (\log K|\xi|)^{1-\alpha} - 1,$$

we obtain

$$\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1](\xi) \geq 2\delta c \frac{(\log K|\xi|)^{1-\alpha} - 1}{(1+K)|\xi|(\log((1+K)|\xi|))^\alpha}$$

for $\xi \in X$. We want to ensure that the left-hand side is not in L^2, so it suffices to show that the right-hand side is not square-integrable. Elementary integration yields

$$\begin{aligned}
\|\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1]\|_{L^2}^2 &= \int_X |\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1](\xi)|^2 \, \mathrm{d}\xi \\
&\geq c \int_{e^L}^\infty \frac{[(\log Kr)^{1-\alpha} - 1]^2}{[\log((1+K)r)]^{2\alpha}} \frac{\mathrm{d}r}{r} \\
&= c \int_L^\infty \frac{[(w + \log K)^{1-\alpha} - 1]^2}{[w + \log(1+K)]^{2\alpha}} \, \mathrm{d}w \\
&\geq c \int_L^\infty \frac{[w/2]^{2(1-\alpha)}}{[2w]^{2\alpha}} \, \mathrm{d}w \\
&= c \int_L^\infty w^{2-4\alpha} \, \mathrm{d}w,
\end{aligned}$$

where $L \geq \max\{\log e/K, \log(1+K)\}$ is chosen sufficiently large such that for all $w > L$, $w^{1-\alpha} - 1 \geq \frac{1}{2} w^{1-\alpha}$. The last integral is finite if and only if $3 - 4\alpha < 0$, i.e. iff $\alpha > 3/4$. Hence, choosing $1/2 < \alpha < 3/4$ ensures that $\nabla \boldsymbol{u} \in H^1$ and $\boldsymbol{B} \in H^1$, but that $\mathscr{F}[((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1] \notin L^2$, and thus that $((\partial_k \boldsymbol{u}) \cdot \nabla)\boldsymbol{B}_1 \notin L^2$.

To complete the counterexample, it only remains to prove Lemmas 5.4 and 5.5.

Proof of Lemma 5.4. Write $\zeta = (r\cos\theta, r\sin\theta)$ and $\eta = (\rho\cos\varphi, \rho\sin\varphi)$. Then

$\theta \in (\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta)$, so

$$\frac{\sqrt{2}}{2} - \delta \leq \cos\theta, \sin\theta \leq \frac{\sqrt{2}}{2} + \delta.$$

Similarly, $\varphi \in (\frac{3\pi}{4} - \delta, \frac{3\pi}{4} + \delta)$, so

$$-\frac{\sqrt{2}}{2} - \delta \leq \cos\varphi \leq -\frac{\sqrt{2}}{2} + \delta,$$
$$\frac{\sqrt{2}}{2} - \delta \leq \sin\varphi \leq \frac{\sqrt{2}}{2} + \delta.$$

Hence

$$\frac{\zeta_1}{|\zeta|} = \cos\theta \geq \frac{\sqrt{2}}{2} - \delta, \qquad \frac{\zeta_2}{|\zeta|} = \sin\theta \geq \frac{\sqrt{2}}{2} - \delta,$$

and

$$\frac{\eta_2}{|\eta|} = \sin\varphi \geq \frac{\sqrt{2}}{2} - \delta.$$

Now,

$$\zeta^{\perp} \cdot \eta = r\rho(\cos\theta \sin\varphi - \sin\theta\cos\varphi) = r\rho\sin(\varphi - \theta).$$

When $\theta \in (\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta)$ and $\varphi \in (\frac{3\pi}{4} - \delta, \frac{3\pi}{4} + \delta)$, we have $\varphi - \theta \in (\frac{\pi}{2} - 2\delta, \frac{\pi}{2} + 2\delta)$. Using the bound $\sin x \geq 1 - \frac{2}{\pi}|x - \frac{\pi}{2}|$ for $x \in (0, \pi)$, we obtain

$$\frac{[\zeta^{\perp} \cdot \eta]}{|\zeta||\eta|} = \sin(\varphi - \theta) \geq 1 - \frac{4\delta}{\pi}.$$

The result follows. $\square$

Proof of Lemma 5.5. First, set

$$S_1 := \{\zeta \in \mathbb{R}^2 : \arg\zeta \in [\tfrac{\pi}{4} - \delta, \tfrac{\pi}{4} + \delta]\},$$
$$S_2 := \{\eta \in \mathbb{R}^2 : \arg\eta \in [\tfrac{3\pi}{4} - \delta, \tfrac{3\pi}{4} + \delta]\},$$

and let $S_3 = \xi - S_2$. Given $\xi \in \Xi$, we seek ζ such that $\zeta \in S_1$ and $\xi - \zeta \in S_2$: to do so, we find the largest $K(\xi)$ such that

$$\{\zeta \in \mathbb{R}^2 : |\zeta| < K(\xi) \text{ and } \arg\zeta \in [\tfrac{\pi}{4} - \delta, \tfrac{\pi}{4} + \delta]\} \subset S_1 \cap S_3.$$

As $\Xi \subset S_2$, S_3 includes zero, and is bounded by the two lines

$$\gamma_1(t) = \xi + t\eta_1, \qquad \gamma_2(t) = \xi + t\eta_2,$$

for $t \geq 0$, where $\eta_1 = -(\cos(\frac{3\pi}{4} + \delta), \sin(\frac{3\pi}{4} + \delta))$, $\eta_2 = -(\cos(\frac{3\pi}{4} - \delta), \sin(\frac{3\pi}{4} - \delta))$.

The line γ_2 has no intersection with S_1, but the line γ_1 will.

It thus suffices to take $K(\xi)$ to be the minimum distance of γ_1 to the origin: let $\xi = r(\cos(\frac{3\pi}{4}+s), \sin(\frac{3\pi}{4}+s))$. Then elementary trigonometry shows that

$$\begin{aligned}
|\gamma_1(t)|^2 &= |\xi + t\eta_1|^2 \\
&= r^2\cos^2(\tfrac{3\pi}{4}+s) - 2rt\cos(\tfrac{3\pi}{4}+s)\cos(\tfrac{3\pi}{4}+\delta) + t^2\cos^2(\tfrac{3\pi}{4}+\delta) \\
&\qquad + r^2\sin^2(\tfrac{3\pi}{4}+s) - 2rt\sin(\tfrac{3\pi}{4}+s)\sin(\tfrac{3\pi}{4}+\delta) + t^2\sin^2(\tfrac{3\pi}{4}+\delta) \\
&= r^2 + t^2 - 2rt[\cos(\tfrac{3\pi}{4}+s)\cos(\tfrac{3\pi}{4}+\delta) + \sin(\tfrac{3\pi}{4}+s)\sin(\tfrac{3\pi}{4}+\delta)] \\
&= r^2 + t^2 - 2rt\cos(\delta - s).
\end{aligned}$$

Differentiating this with respect to t, we see that $|\gamma_1(t)|^2$ is minimised when $t = r\cos(\delta - s)$, whence

$$|\gamma_1(t)|^2 \geq r^2(1 - \cos^2(\delta - s)) = r^2\sin^2(\delta - s).$$

Since $s \in [-\frac{\delta}{2}, \frac{\delta}{2}]$, $\delta - s \in [\frac{\delta}{2}, \frac{3\delta}{2}]$. Hence $|\gamma_1(t)| \geq |\xi|\sin\frac{\delta}{2}$, meaning that $\Upsilon_\xi \subset S_1 \cap S_3$, so choosing $\xi \in \Xi$ and $\zeta \in \Upsilon_\xi$ guarantees that $\xi - \zeta \in S_2$, as required. $\square$

Chapter 6

Local Existence in Sobolev Spaces for Non-Resistive MHD and Stokes-MHD

6.1 Local Existence for Non-Resistive MHD

We return to the MHD equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \tag{1.5a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \tag{1.5b}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0 \tag{1.5c}$$

on the whole of $\mathbb{R}^n$, with initial data $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$ satisfying $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$, for $s > n/2$. We will prove the following theorem.

Theorem 6.1. *Let $n = 2, 3$. For $s > n/2$, and initial data $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$ with $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$, there exists a time $T_* = T_*(s, \|\boldsymbol{u}_0\|_{H^s}, \|\boldsymbol{B}_0\|_{H^s}) > 0$ such that the equations* (1.5) *have a unique solution* $(\boldsymbol{u}, \boldsymbol{B})$*, with*

$$\begin{aligned} \boldsymbol{u} &\in C([0, T_*]; H^s(\mathbb{R}^n)) \cap L^2(0, T_*; H^{s+1}(\mathbb{R}^n)), \\ \boldsymbol{B} &\in C([0, T_*]; H^s(\mathbb{R}^n)). \end{aligned}$$

The proof depends fundamentally on the commutator estimate in Theorem 5.1, which requires $s > n/2$. We have seen in Section 5.2 that the commutator estimate is not valid for $s = 1$ in 2D; this therefore suggests that proving local existence in $H^{n/2}$ (if possible) would require a more refined technique. We note that in a recent paper Bourgain & Li (2013) showed that the Euler equations on $\mathbb{R}^n$ are in fact ill-posed in $H^{1+n/2}$ $(n = 2, 3)$; in light of this it seems likely that system (1.5) is ill-posed in $H^{n/2}$.

The general strategy of the proof is similar to that for proving existence of solutions to the Navier–Stokes and Euler equations which can be found in Section 3.2 of Majda & Bertozzi (2002), for example. First, we show that the solutions $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ of some smoothed version of the equations exist and are uniformly bounded in H^s. We then show they are Cauchy in the L^2 norm as $R \to \infty$. By interpolation, $(\boldsymbol{u}^R, \boldsymbol{B}^R) \to (\boldsymbol{u}, \boldsymbol{B})$ in any $H^{s'}$ for $0 < s' < s$, which implies that $(\boldsymbol{u}, \boldsymbol{B})$ solve the original equations.

As in Section 4.2.2, we define the Fourier truncation $\mathcal{S}_R$ as follows:

$$\widehat{\mathcal{S}_R f}(\xi) = \mathbb{1}_{B_R}(\xi)\hat{f}(\xi),$$

where B_R denotes the ball of radius R centered at the origin. Note that

$$\begin{aligned}
\|\mathcal{S}_R f - f\|_{H^s}^2 &= \int_{(B_R)^c} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\
&= \int_{(B_R)^c} \frac{1}{(1 + |\xi|^2)^k} (1 + |\xi|^2)^{s+k} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\
&\leq \frac{1}{(1 + R^2)^k} \int_{(B_R)^c} (1 + |\xi|^2)^{s+k} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\
&\leq \frac{C}{R^{2k}} \|f\|_{H^{s+k}}^2.
\end{aligned}$$

Hence

$$\|\mathcal{S}_R f - f\|_{H^s} \leq C(1/R)^k \|f\|_{H^{s+k}}, \tag{6.1}$$

$$\|\mathcal{S}_R f - \mathcal{S}_{R'} f\|_{H^s} \leq C \max\{(1/R)^k, (1/R')^k\} \|f\|_{H^{s+k}}. \tag{6.2}$$

We consider the truncated MHD equations on the whole of $\mathbb{R}^n$:

$$\frac{\partial \boldsymbol{u}^R}{\partial t} - \nu \Delta \boldsymbol{u}^R + \nabla p_*^R = \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R] - \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R], \tag{6.3a}$$

$$\frac{\partial \boldsymbol{B}^R}{\partial t} = \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] - \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R], \tag{6.3b}$$

$$\nabla \cdot \boldsymbol{u}^R = \nabla \cdot \boldsymbol{B}^R = 0, \tag{6.3c}$$

with initial data $\mathcal{S}_R\boldsymbol{u}_0, \mathcal{S}_R\boldsymbol{B}_0$. By taking the truncated initial data as we have, we ensure that $\boldsymbol{u}^R, \boldsymbol{B}^R$ lie in the space

$$V_R := \{f \in L^2(\mathbb{R}^n) : \hat{f} \text{ is supported in } B_R\},$$

as the truncations are invariant under the flow of the equations. The Fourier truncations act like mollifiers, smoothing the equation; in particular, on the space V_R it is easy to show that

$$F(\boldsymbol{u}^R, \boldsymbol{B}^R) := \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R]$$

is Lipschitz in $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$. Hence, by Picard's theorem for infinite-dimensional ODEs (see Theorem 3.1 in Majda & Bertozzi (2002), for example), there exists a solution $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ in V_R to (6.3) for some time interval $[0, T(R)]$. The solution will exist as long as $\|\boldsymbol{u}^R\|_{H^s}$ and $\|\boldsymbol{B}^R\|_{H^s}$ remain finite.

Proposition 6.2. *Given initial data $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$ with $s > n/2$, there exists a time T_* such that the quantities*

$$\sup_{t\in[0,T_*]} \|\boldsymbol{u}^R(t)\|_{H^s}, \quad \sup_{t\in[0,T_*]} \|\boldsymbol{B}^R(t)\|_{H^s}, \quad \int_0^{T_*} \|\nabla \boldsymbol{u}^R(t)\|_{H^s}^2 \,\mathrm{d}t$$

are bounded uniformly in R.

Before embarking on the proof, we first prove a simple energy estimate: take the inner product of (6.3a) with $\boldsymbol{u}^R$ and the inner product of (6.3b) with $\boldsymbol{B}^R$, and add to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}^R\|_{L^2}^2 + \|\boldsymbol{B}^R\|_{L^2}^2) + \nu\|\nabla \boldsymbol{u}^R\|_{L^2}^2 = 0; \tag{6.4}$$

integrating and using the fact that $\|\boldsymbol{u}^R(0)\|_{L^2} \le \|\boldsymbol{u}_0\|_{L^2}$ and $\|\boldsymbol{B}^R(0)\|_{L^2} \le \|\boldsymbol{B}_0\|_{L^2}$ yields

$$\|\boldsymbol{u}^R(t)\|_{L^2}^2 + \|\boldsymbol{B}^R(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \boldsymbol{u}^R(s)\|_{L^2}^2 \,\mathrm{d}s \le \|\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{B}_0\|_{L^2}^2. \tag{6.5}$$

Proof of Proposition 6.2. For $s > n/2$, apply Λ^s to both equations:

$$\frac{\partial}{\partial t}\Lambda^s \boldsymbol{u}^R - \nu\Delta\Lambda^s \boldsymbol{u}^R + \nabla\Lambda^s p_*^R = \mathcal{S}_R\Lambda^s[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R] - \mathcal{S}_R\Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R],$$
$$\frac{\partial}{\partial t}\Lambda^s \boldsymbol{B}^R = \mathcal{S}_R\Lambda^s[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] - \mathcal{S}_R\Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R].$$

Take the inner product of the first equation with $\Lambda^s \boldsymbol{u}^R$, and the inner product of the second equation with $\Lambda^s \boldsymbol{B}^R$, to obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s \boldsymbol{u}^R\|_{L^2}^2 + \nu\|\Lambda^s \nabla \boldsymbol{u}^R\|_{L^2}^2 &= \langle \Lambda^s[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R], \Lambda^s \boldsymbol{u}^R\rangle \\
&\quad - \langle \Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R], \Lambda^s \boldsymbol{u}^R\rangle, \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s \boldsymbol{B}^R\|_{L^2}^2 &= \langle \Lambda^s[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R], \Lambda^s \boldsymbol{B}^R\rangle \\
&\quad - \langle \Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R], \Lambda^s \boldsymbol{B}^R\rangle.
\end{aligned}$$

Note that we have used the fact that $\mathcal{S}_R \boldsymbol{u}^R = \boldsymbol{u}^R$, since $\boldsymbol{u}^R \in V_R$.

The most difficult term, $\langle \Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R], \Lambda^s \boldsymbol{B}^R\rangle$, is dealt with easily by our commutator estimate (Corollary 5.3):

$$\left|\langle \Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R], \Lambda^s \boldsymbol{B}^R\rangle\right| \le c\|\nabla \boldsymbol{u}^R\|_{H^s}\|\boldsymbol{B}^R\|_{H^s}^2.$$

The other three terms can be estimated using the fact that H^s is an algebra for $s > n/2$. Two follow directly:

$$\begin{aligned}
\left|\langle \Lambda^s[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R], \Lambda^s \boldsymbol{u}^R\rangle\right| &\le c\|\nabla \boldsymbol{u}^R\|_{H^s}\|\boldsymbol{u}^R\|_{H^s}^2, \\
\left|\langle \Lambda^s[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R], \Lambda^s \boldsymbol{B}^R\rangle\right| &\le c\|\nabla \boldsymbol{u}^R\|_{H^s}\|\boldsymbol{B}^R\|_{H^s}^2,
\end{aligned}$$

while the remaining term requires an integration by parts:

$$\begin{aligned}
\left|\langle \Lambda^s[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R], \Lambda^s \boldsymbol{u}^R\rangle\right| &= \left| -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \Lambda^s[\boldsymbol{B}_i^R \boldsymbol{B}_j^R]\Lambda^s \partial_i \boldsymbol{u}_j^R \right| \\
&\le c\|\boldsymbol{B}^R\|_{H^s}^2\|\nabla \boldsymbol{u}^R\|_{H^s}.
\end{aligned}$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}^R\|_{\dot{H}^s}^2 + \|\boldsymbol{B}^R\|_{\dot{H}^s}^2) + \nu\|\nabla \boldsymbol{u}^R\|_{\dot{H}^s}^2 \le c\|\nabla \boldsymbol{u}^R\|_{H^s}(\|\boldsymbol{u}^R\|_{H^s}^2 + \|\boldsymbol{B}^R\|_{H^s}^2).$$

Combining this with the energy estimate (6.4) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}^R\|_{H^s}^2 + \|\boldsymbol{B}^R\|_{H^s}^2) + \nu\|\nabla \boldsymbol{u}^R\|_{H^s}^2 \le c\|\nabla \boldsymbol{u}^R\|_{H^s}(\|\boldsymbol{u}^R\|_{H^s}^2 + \|\boldsymbol{B}^R\|_{H^s}^2).$$

By Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}^R\|_{H^s}^2 + \|\boldsymbol{B}^R\|_{H^s}^2) + \nu\|\nabla \boldsymbol{u}^R\|_{H^s}^2 \le \frac{c}{\nu}(\|\boldsymbol{u}^R\|_{H^s}^2 + \|\boldsymbol{B}^R\|_{H^s}^2)^2.$$

Setting $Y(t) = (\|\boldsymbol{u}^R\|_{H^s}^2 + \|\boldsymbol{B}^R\|_{H^s}^2)$ and $Y_0 = (\|\boldsymbol{u}_0\|_{H^s}^2 + \|\boldsymbol{B}_0\|_{H^s}^2)$, a standard Gronwall-type argument shows that

$$Y(t) \leq \frac{\nu Y_0}{\nu - CTY_0} \tag{6.6}$$

for all $t \in [0, T]$. So provided we choose $T_* < CY_0/\nu$, $\|\boldsymbol{u}^R\|_{H^s}$ and $\|\boldsymbol{B}^R\|_{H^s}$ remain bounded on $[0, T_*]$ independently of R, and $\int_0^{T_*} \|\nabla \boldsymbol{u}^R(t)\|_{H^s}^2 \, \mathrm{d}t$ is bounded uniformly in R. □

Having proven these uniform estimates, we could use the Aubin–Lions compactness theorem (Theorem 4.5) to extract a subsequence $(\boldsymbol{u}^{R_m}, \boldsymbol{B}^{R_m})$ that converges strongly to $(\boldsymbol{u}, \boldsymbol{B})$ in some sense; while this approach is natural when working on a bounded domain, on the whole space one only obtains the requisite strong convergence on compact subsets, and one must then appeal (as in Section 4.2.2) to the argument of, for example, Chemin et al. (2006), §2.2.4, to show that, indeed, the nonlinear terms converge as required.

While such an approach is natural when working with weak solutions, when dealing with strong solutions (as we are here) it is significantly simpler to follow the approach of, for example, Majda & Bertozzi (2002) and show that $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ converge strongly in $L^\infty(0, T_*; L^2(\mathbb{R}^n))$, by showing they are Cauchy as $R \to \infty$.

Proposition 6.3. *The family $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ of solutions of (6.3) is Cauchy (as $R \to \infty$) in $L^\infty(0, T; L^2(\mathbb{R}^n))$.*

Proof. Consider the equations

$$\frac{\partial \boldsymbol{u}^R}{\partial t} - \nu \Delta \boldsymbol{u}^R + \nabla p_*^R = \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R] - \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R], \tag{6.7a}$$

$$\frac{\partial \boldsymbol{B}^R}{\partial t} = \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] - \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R], \tag{6.7b}$$

$$\nabla \cdot \boldsymbol{u}^R = \nabla \cdot \boldsymbol{B}^R = 0. \tag{6.7c}$$

Take the difference between the equations for R and R':

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{u}^R - \boldsymbol{u}^{R'}) - \nu \Delta \left(\boldsymbol{u}^R - \boldsymbol{u}^{R'}\right) &+ \nabla(p_*^R - p_*^{R'}) \\ &= \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R] - \mathcal{S}_{R'}[(\boldsymbol{B}^{R'} \cdot \nabla)\boldsymbol{B}^{R'}] \\ &\quad - \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R] + \mathcal{S}_{R'}[(\boldsymbol{u}^{R'} \cdot \nabla)\boldsymbol{u}^{R'}], \end{aligned} \tag{6.8a}$$

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{B}^R - \boldsymbol{B}^{R'}) &= \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] - \mathcal{S}_{R'}[(\boldsymbol{B}^{R'} \cdot \nabla)\boldsymbol{u}^{R'}] \\ &\quad - \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R] + \mathcal{S}_{R'}[(\boldsymbol{u}^{R'} \cdot \nabla)\boldsymbol{B}^{R'}], \end{aligned} \tag{6.8b}$$

Take the inner product of (6.8a) with $\boldsymbol{u}^R - \boldsymbol{u}^{R'}$ and the inner product of (6.8b) with $\boldsymbol{B}^R - \boldsymbol{B}^{R'}$ and add to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}^2 + \|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2}^2\right) + \nu\|\nabla(\boldsymbol{u}^R - \boldsymbol{u}^{R'})\|_{L^2}^2 \\
&= \langle \mathcal{S}_R[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R] - \mathcal{S}_{R'}[(\boldsymbol{B}^{R'}\cdot\nabla)\boldsymbol{B}^{R'}], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.9a)}\\
&\quad - \langle \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R] - \mathcal{S}_{R'}[(\boldsymbol{u}^{R'}\cdot\nabla)\boldsymbol{u}^{R'}], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.9b)}\\
&\quad + \langle \mathcal{S}_R[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{u}^R] - \mathcal{S}_{R'}[(\boldsymbol{B}^{R'}\cdot\nabla)\boldsymbol{u}^{R'}], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && \text{(6.9c)}\\
&\quad - \langle \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R] - \mathcal{S}_{R'}[(\boldsymbol{u}^{R'}\cdot\nabla)\boldsymbol{B}^{R'}], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && \text{(6.9d)}
\end{aligned}$$

Now, fix $0 < \varepsilon < s - 1$ and consider (6.9b). We split it into three terms:

$$\begin{aligned}
&\langle \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R] - \mathcal{S}_{R'}[(\boldsymbol{u}^{R'}\cdot\nabla)\boldsymbol{u}^{R'}], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle \\
&\quad = \langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.10a)}\\
&\quad + \langle \mathcal{S}_{R'}[((\boldsymbol{u}^R - \boldsymbol{u}^{R'})\cdot\nabla)\boldsymbol{u}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.10b)}\\
&\quad + \langle \mathcal{S}_{R'}[(\boldsymbol{u}^{R'}\cdot\nabla)(\boldsymbol{u}^R - \boldsymbol{u}^{R'})], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.10c)}
\end{aligned}$$

Notice that (6.10c) is zero (just integrate by parts and use the divergence-free condition). For (6.10b) we have

$$\begin{aligned}
\left|\langle \mathcal{S}_{R'}[((\boldsymbol{u}^R - \boldsymbol{u}^{R'})\cdot\nabla)\boldsymbol{u}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle\right| &\le c\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}^2\|\nabla\boldsymbol{u}^R\|_{L^\infty}\\
&\le c\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}^2\|\nabla\boldsymbol{u}^R\|_{H^s}.
\end{aligned}$$

For (6.10a) we use estimate (6.2) (recalling that $R' > R$) to obtain

$$\begin{aligned}
\left|\langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle\right| &\le \frac{1}{R^\varepsilon}\|(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R\|_{H^\varepsilon}\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}\\
&= \frac{1}{R^\varepsilon}\|\nabla\cdot(\boldsymbol{u}^R\otimes\boldsymbol{u}^R)\|_{H^\varepsilon}\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}\\
&\le \frac{1}{R^\varepsilon}\|(\boldsymbol{u}^R\otimes\boldsymbol{u}^R)\|_{H^s}\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}\\
&\le \frac{1}{R^\varepsilon}\|\boldsymbol{u}^R\|_{H^s}^2\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}.
\end{aligned}$$

We now consider (6.9d), and again split it into three terms:

$$\begin{aligned}
&\langle \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R] - \mathcal{S}_{R'}[(\boldsymbol{u}^{R'}\cdot\nabla)\boldsymbol{B}^{R'}], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle \\
&\qquad = \langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && (6.11\text{a})\\
&\qquad + \langle \mathcal{S}_{R'}[((\boldsymbol{u}^R - \boldsymbol{u}^{R'})\cdot\nabla)\boldsymbol{B}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && (6.11\text{b})\\
&\qquad + \langle \mathcal{S}_{R'}[(\boldsymbol{u}^{R'}\cdot\nabla)(\boldsymbol{B}^R - \boldsymbol{B}^{R'})], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && (6.11\text{c})
\end{aligned}$$

Again, (6.11c) is zero, and we may treat (6.11a) as we treated (6.10a) to obtain

$$\left|\langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle\right| \le \frac{1}{R^\varepsilon}\|\boldsymbol{u}^R\|_{H^s}\|\boldsymbol{B}^R\|_{H^s}\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2}.$$

However, (6.11b) is a bit more delicate and requires more care: in particular it requires different treatments in two and three dimensions: in 2D, we use

$$\|fg\|_{L^2} \le \|f\|_{L^{2/\varepsilon}}\|g\|_{L^{2/(1+\varepsilon)}} \le c\|f\|_{H^{1-\varepsilon}}\|g\|_{H^\varepsilon} \le c\|f\|_{H^1}\|g\|_{H^{s-1}},$$

while in 3D we use

$$\|fg\|_{L^2} \le \|f\|_{L^6}\|g\|_{L^3} \le c\|f\|_{H^1}\|g\|_{H^{1/2}} \le c\|f\|_{H^1}\|g\|_{H^{s-1}}.$$

In either case, we obtain

$$\begin{aligned}
&\left|\langle \mathcal{S}_{R'}[((\boldsymbol{u}^R - \boldsymbol{u}^{R'})\cdot\nabla)\boldsymbol{B}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle\right| \\
&\qquad = \left|\langle ((\boldsymbol{u}^R - \boldsymbol{u}^{R'})\cdot\nabla)\boldsymbol{B}^R, \mathcal{S}_{R'}[\boldsymbol{B}^R - \boldsymbol{B}^{R'}]\rangle\right| \\
&\qquad \le c\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{H^1}\|\nabla\boldsymbol{B}^R\|_{H^{s-1}}\|\mathcal{S}_{R'}[\boldsymbol{B}^R - \boldsymbol{B}^{R'}]\|_{L^2}.
\end{aligned}$$

We now use the inequality $ab \le \frac{1}{\nu}a^2 + \frac{\nu}{4}b^2$, yielding

$$(6.11\text{b}) \le \frac{\nu}{4}\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{H^1}^2 + \frac{c}{\nu}\|\nabla\boldsymbol{B}^R\|_{H^{s-1}}^2\|\mathcal{S}_{R'}[\boldsymbol{B}^R - \boldsymbol{B}^{R'}]\|_{L^2}^2,$$

We consider the last two terms, (6.9a) and (6.9c) together, splitting them into six terms in all:

$$\begin{aligned}
&\langle \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R] - \mathcal{S}_{R'}[(\boldsymbol{B}^{R'} \cdot \nabla)\boldsymbol{B}^{R'}], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle \\
&+ \langle \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] - \mathcal{S}_{R'}[(\boldsymbol{B}^{R'} \cdot \nabla)\boldsymbol{u}^{R'}], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle \\
&\quad = \langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.12a)} \\
&\quad + \langle \mathcal{S}_{R'}[((\boldsymbol{B}^R - \boldsymbol{B}^{R'}) \cdot \nabla)\boldsymbol{B}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.12b)} \\
&\quad + \langle \mathcal{S}_{R'}[(\boldsymbol{B}^{R'} \cdot \nabla)(\boldsymbol{B}^R - \boldsymbol{B}^{R'})], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle && \text{(6.12c)} \\
&\quad + \langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && \text{(6.12d)} \\
&\quad + \langle \mathcal{S}_{R'}[((\boldsymbol{B}^R - \boldsymbol{B}^{R'}) \cdot \nabla)\boldsymbol{u}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && \text{(6.12e)} \\
&\quad + \langle \mathcal{S}_{R'}[(\boldsymbol{B}^{R'} \cdot \nabla)(\boldsymbol{u}^R - \boldsymbol{u}^{R'})], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle && \text{(6.12f)}
\end{aligned}$$

As before, (6.12c) and (6.12f) add to zero. We may treat (6.12a) and (6.12d) as we treated (6.10a) and (6.11a) to obtain

$$\begin{aligned}
\left|\langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle\right| &\le \frac{1}{R^\varepsilon}\|\boldsymbol{B}^R\|_{H^s}^2 \|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{L^2}, \\
\left|\langle (\mathcal{S}_R - \mathcal{S}_{R'})[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle\right| &\le \frac{1}{R^\varepsilon}\|\boldsymbol{B}^R\|_{H^s} \|\boldsymbol{u}^R\|_{H^s} \|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2}.
\end{aligned}$$

Of the two remaining terms, (6.12e) can be dealt with in the same way as (6.10b):

$$\begin{aligned}
\left|\langle \mathcal{S}_{R'}[((\boldsymbol{B}^R - \boldsymbol{B}^{R'}) \cdot \nabla)\boldsymbol{u}^R], \boldsymbol{B}^R - \boldsymbol{B}^{R'}\rangle\right| &\le c\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2}^2 \|\nabla \boldsymbol{u}^R\|_{L^\infty} \\
&\le c\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2}^2 \|\nabla \boldsymbol{u}^R\|_{H^s}.
\end{aligned}$$

To bound (6.12b), we first integrate by parts to obtain

$$\begin{aligned}
&\left|\langle \mathcal{S}_{R'}[((\boldsymbol{B}^R - \boldsymbol{B}^{R'}) \cdot \nabla)\boldsymbol{B}^R], \boldsymbol{u}^R - \boldsymbol{u}^{R'}\rangle\right| \\
&\quad = \left|\langle ((\boldsymbol{B}^R - \boldsymbol{B}^{R'}) \cdot \nabla)\boldsymbol{B}^R, \mathcal{S}_{R'}[\boldsymbol{u}^R - \boldsymbol{u}^{R'}]\rangle\right| \\
&\quad = \left|\langle ((\boldsymbol{B}^R - \boldsymbol{B}^{R'}) \cdot \nabla)\mathcal{S}_{R'}[\boldsymbol{u}^R - \boldsymbol{u}^{R'}], \boldsymbol{B}^R\rangle\right| \\
&\quad \le \|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2} \|\nabla \mathcal{S}_{R'}[\boldsymbol{u}^R - \boldsymbol{u}^{R'}]\|_{L^2} \|\boldsymbol{B}^R\|_{L^\infty} \\
&\quad \le c\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2} \|\nabla \mathcal{S}_{R'}[\boldsymbol{u}^R - \boldsymbol{u}^{R'}]\|_{L^2} \|\boldsymbol{B}^R\|_{H^s}.
\end{aligned}$$

As in (6.11b), we use the inequality $ab \le \frac{1}{\nu}a^2 + \frac{\nu}{4}b^2$ to obtain

$$\text{(6.12b)} \le \frac{c}{\nu}\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{L^2}^2 \|\boldsymbol{B}^R\|_{H^s}^2 + \frac{\nu}{4}\|\nabla \mathcal{S}_{R'}[\boldsymbol{u}^R - \boldsymbol{u}^{R'}]\|_{L^2}^2.$$

Putting all the terms together we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}^2+\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}^2\right)+\nu\|\nabla(\boldsymbol{u}^R-\boldsymbol{u}^{R'})\|_{L^2}^2\\
&\quad=\frac{1}{R^\varepsilon}\|\boldsymbol{B}^R\|_{H^s}^2\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}+\frac{c}{\nu}\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}^2\|\boldsymbol{B}^R\|_{H^s}^2\\
&\qquad+\frac{1}{R^\varepsilon}\|\boldsymbol{u}^R\|_{H^s}^2\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}+c\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}^2\|\nabla\boldsymbol{u}^R\|_{H^s}\\
&\qquad+\frac{1}{R^\varepsilon}\|\boldsymbol{B}^R\|_{H^s}\|\boldsymbol{u}^R\|_{H^s}\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}+c\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}^2\|\nabla\boldsymbol{u}^R\|_{H^s}\\
&\qquad+\frac{1}{R^\varepsilon}\|\boldsymbol{u}^R\|_{H^s}\|\boldsymbol{B}^R\|_{H^s}\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}+\frac{c}{\nu}\|\boldsymbol{B}^R\|_{H^s}^2\|\mathcal{S}_{R'}[\boldsymbol{B}^R-\boldsymbol{B}^{R'}]\|_{L^2}^2\\
&\quad\le\frac{1}{R^\varepsilon}\left(\|\boldsymbol{u}^R\|_{H^s}^2+\|\boldsymbol{B}^R\|_{H^s}^2\right)\left(\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}+\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}\right)\\
&\qquad+c\|\nabla\boldsymbol{u}^R\|_{H^s}\left(\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}^2+\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}^2\right)\\
&\qquad+\frac{c}{\nu}\|\boldsymbol{B}^R\|_{H^s}^2\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}^2.
\end{aligned}
$$

Setting $Y(t)=\|\boldsymbol{u}^R-\boldsymbol{u}^{R'}\|_{L^2}+\|\boldsymbol{B}^R-\boldsymbol{B}^{R'}\|_{L^2}$, and using the bound

$$
\sup_{t\in[0,T_*]}\|\boldsymbol{u}^R(t)\|_{H^s},\ \sup_{t\in[0,T_*]}\|\boldsymbol{B}^R(t)\|_{H^s},\ \int_0^{T_*}\|\nabla\boldsymbol{u}^R(t)\|_{H^s}^2\,\mathrm{d}t\le M
$$

for all $t\in[0,T_*]$, we see that

$$
\frac{\mathrm{d}Y}{\mathrm{d}t}\le\frac{M}{R^\varepsilon}+cY\left(\frac{M}{\nu}+\|\nabla\boldsymbol{u}^R\|_{H^s}\right).
$$

As $\|\nabla\boldsymbol{u}^R\|_{H^s}$ is integrable in time, a standard Gronwall argument shows that

$$
\sup_{t\in[0,T_*]}Y(t)\le\frac{C(\nu,M,T_*)}{R^\varepsilon},
$$

and the right-hand side tends to zero as as $R,R'\to\infty$, as required. □

It follows that $(\boldsymbol{u}^R,\boldsymbol{B}^R)\to(\boldsymbol{u},\boldsymbol{B})$ strongly in $L^\infty(0,T_*;L^2(\mathbb{R}^n))$; it is straightforward to use the last estimate in the proof above to show that $\nabla\boldsymbol{u}^R\to\nabla\boldsymbol{u}$ strongly in $L^2(0,T_*;L^2(\mathbb{R}^n))$. We now combine Propositions 6.2 and 6.3 together with the following standard interpolation lemma (see Adams & Fournier (2003) for the proof).

Lemma 6.4. *Given $s>0$, there exists a constant C_s such that, for all $\boldsymbol{v}\in H^s(\mathbb{R}^n)$ and all $0<s'<s$,*

$$
\|\boldsymbol{v}\|_{H^{s'}}\le C_s\|\boldsymbol{v}\|_{L^2}^{1-s'/s}\|\boldsymbol{v}\|_{H^s}^{s'/s}.
$$

Using Lemma 6.4 gives

$$\sup_{t\in[0,T_*]}(\|\boldsymbol{u}^R-\boldsymbol{u}\|_{H^{s'}}+\|\boldsymbol{B}^R-\boldsymbol{B}\|_{H^{s'}})\leq C(T_*,\|\boldsymbol{u}_0\|_{H^s},\|\boldsymbol{B}_0\|_{H^s})\left(\frac{1}{R^\varepsilon}\right)^{1-s'/s};$$

in other words, $(\boldsymbol{u}^R,\boldsymbol{B}^R)\to(\boldsymbol{u},\boldsymbol{B})$ strongly in $L^\infty(0,T_*;H^{s'}(\mathbb{R}^n))$ for any $s'<s$. Furthermore, $\nabla\boldsymbol{u}^R\to\nabla\boldsymbol{u}$ strongly in $L^2(0,T_*;H^{s'}(\mathbb{R}^n))$ for any $s'<s$, and thus $\Delta\boldsymbol{u}^R\to\Delta\boldsymbol{u}$ strongly in $L^2(0,T_*;H^{s'-1}(\mathbb{R}^n))$. To deal with the nonlinear terms, we prove a simple estimate.

Lemma 6.5. *Fix $s>n/2$ and let $\boldsymbol{v},\boldsymbol{w}\in H^s$ with $\nabla\cdot\boldsymbol{v}=0$. Then*

$$\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\|_{H^{s-1}}\leq C\|\boldsymbol{v}\|_{H^s}\|\boldsymbol{w}\|_{H^s}.$$

Proof. As $\boldsymbol{v}$ is divergence-free, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}=\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{w})$. As H^s is an algebra,

$$\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\|_{H^{s-1}}=\|\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{w})\|_{H^{s-1}}\leq C\|\boldsymbol{v}\otimes\boldsymbol{w}\|_{H^s}\leq C\|\boldsymbol{v}\|_{H^s}\|\boldsymbol{w}\|_{H^s}.\qquad\square$$

For $s'>n/2$, by Lemma 6.5,

$$\sup_{t\in[0,T_*]}\|\mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R]-(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\|_{H^{s'-1}}\to 0$$

as $R\to\infty$. It remains to show convergence of the time derivatives: using Lemma 6.5 once more, we obtain

$$\left\|\frac{\partial\boldsymbol{u}^R}{\partial t}\right\|_{H^{s-1}}+\left\|\frac{\partial\boldsymbol{B}^R}{\partial t}\right\|_{H^{s-1}}\leq C\|\Delta\boldsymbol{u}^R\|_{H^{s-1}}+C\left(\|\boldsymbol{u}^R\|_{H^s}+\|\boldsymbol{B}^R\|_{H^s}\right)^2.$$

Using this and Proposition 6.2, we can extract a subsequence $R_m\to+\infty$ such that

$$\frac{\partial\boldsymbol{u}^{R_m}}{\partial t}\overset{*}{\rightharpoonup}\frac{\partial\boldsymbol{u}}{\partial t},\frac{\partial\boldsymbol{B}^{R_m}}{\partial t}\overset{*}{\rightharpoonup}\frac{\partial\boldsymbol{B}}{\partial t}\text{ in }L^2(0,T_*;H^{s-1}(\mathbb{R}^n)).$$

Using the above strong convergence allows us to conclude that the time derivatives will converge strongly in $L^2(0,T_*;H^{s'-1}(\mathbb{R}^n))$ as well, and hence $(\boldsymbol{u},\boldsymbol{B})$ solves (1.5) as an equality in $L^2(0,T_*;H^{s'-1}(\mathbb{R}^n))$. Finally, the uniform bounds in Proposition 6.2 guarantee the existence of a subsequence (which we relabel) such that

$$\begin{aligned}\boldsymbol{u}^{R_m}\overset{*}{\rightharpoonup}\boldsymbol{u},\boldsymbol{B}^{R_m}\overset{*}{\rightharpoonup}\boldsymbol{B}\text{ in }L^\infty(0,T_*;H^s(\mathbb{R}^n)),\\ \nabla\boldsymbol{u}^{R_m}\overset{*}{\rightharpoonup}\nabla\boldsymbol{u}\text{ in }L^2(0,T_*;H^s(\mathbb{R}^n))\end{aligned}$$

(by the Banach–Alaoglu theorem), which guarantees that the limit satisfies

$$\boldsymbol{u} \in L^\infty(0,T_*;H^s(\mathbb{R}^n)) \cap L^2(0,T_*;H^{s+1}(\mathbb{R}^n)), \qquad \boldsymbol{B} \in L^\infty(0,T_*;H^s(\mathbb{R}^n)).$$

This completes the proof of existence in Theorem 6.1. We now prove uniqueness.

Proposition 6.6. *Let* $(\boldsymbol{u}_j, \boldsymbol{B}_j)$, $j = 1, 2$, *be two solutions of* (1.5) *with the same initial conditions* $\boldsymbol{u}_j(0) = \boldsymbol{u}_0$, $\boldsymbol{B}_j(0) = \boldsymbol{B}_0$, *such that*

$$\begin{aligned} &\boldsymbol{u}_j \in L^\infty(0,T_*;H^s(\mathbb{R}^n)) \cap L^2(0,T_*;H^{s+1}(\mathbb{R}^n)), \\ &\boldsymbol{B}_j \in L^\infty(0,T_*;H^s(\mathbb{R}^n)). \end{aligned}$$

Then $(\boldsymbol{u}_1, \boldsymbol{B}_1) = (\boldsymbol{u}_2, \boldsymbol{B}_2)$ *as functions in* $L^\infty(0,T;L^2(\mathbb{R}^n))$.

Proof. Take the equations for $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2)$ and subtract: writing $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{z} = \boldsymbol{B}_1 - \boldsymbol{B}_2$ and $q = p_1 - p_2$, we obtain

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_2 - \nu\Delta\boldsymbol{w} + \nabla q = (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{z} + (\boldsymbol{z} \cdot \nabla)\boldsymbol{B}_2, \tag{6.13a}$$

$$\frac{\partial \boldsymbol{z}}{\partial t} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{z} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{B}_2 = (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{w} + (\boldsymbol{z} \cdot \nabla)\boldsymbol{u}_2. \tag{6.13b}$$

Taking the inner product of (6.13a) with $\boldsymbol{w}$ and (6.13b) with $\boldsymbol{z}$ and adding yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{w}\|_{L^2}^2 + \|\boldsymbol{z}\|_{L^2}^2\right) + \nu\|\nabla\boldsymbol{w}\|_{L^2}^2 \\ &\qquad = \langle(\boldsymbol{z} \cdot \nabla)\boldsymbol{B}_2, \boldsymbol{w}\rangle - \langle(\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_2, \boldsymbol{w}\rangle + \langle(\boldsymbol{z} \cdot \nabla)\boldsymbol{u}_2, \boldsymbol{z}\rangle - \langle(\boldsymbol{w} \cdot \nabla)\boldsymbol{B}_2, \boldsymbol{z}\rangle \\ &\qquad \le \|\boldsymbol{z}\|_{L^2}\|\nabla\boldsymbol{w}\|_{L^2}\|\boldsymbol{B}_2\|_{L^\infty} + \|\boldsymbol{w}\|_{L^2}\|\nabla\boldsymbol{w}\|_{L^2}\|\boldsymbol{u}_2\|_{L^\infty} \\ &\qquad\qquad + \|\boldsymbol{z}\|_{L^2}^2\|\nabla\boldsymbol{u}_2\|_{L^\infty} + \|\boldsymbol{w}\|_{L^{2/\varepsilon}}\|\nabla\boldsymbol{B}_2\|_{L^{2/(1-\varepsilon)}}\|\boldsymbol{z}\|_{L^2} \\ &\qquad \le \left(\|\boldsymbol{w}\|_{L^2} + \|\boldsymbol{z}\|_{L^2}\right)\|\nabla\boldsymbol{w}\|_{L^2}\left(\|\boldsymbol{u}_2\|_{H^s} + \|\boldsymbol{B}_2\|_{H^s}\right) + \|\boldsymbol{z}\|_{L^2}^2\|\nabla\boldsymbol{u}_2\|_{H^s}, \end{aligned}$$

so by Young's inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{w}\|_{L^2}^2 + \|\boldsymbol{z}\|_{L^2}^2\right) + \nu\|\nabla\boldsymbol{w}\|_{L^2}^2 \le \left(M + \|\nabla\boldsymbol{u}_2\|_{H^s}\right)\left(\|\boldsymbol{w}\|_{L^2}^2 + \|\boldsymbol{z}\|_{L^2}^2\right)$$

and uniqueness follows by Gronwall's inequality. □

As $\boldsymbol{u} \in L^2(0,T_*;H^{s+1}(\mathbb{R}^n))$ and $\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T_*;H^{s-1}(\mathbb{R}^n))$, by standard results (see, e.g., Evans (1998), §5.9, Theorem 4), $\boldsymbol{u} \in C([0,T_*];H^s(\mathbb{R}^n))$. However, a further argument is needed to show that $\boldsymbol{B} \in C([0,T_*];H^s(\mathbb{R}^n))$: we proceed as in Theorem 3.5 (pp109–111) in Majda & Bertozzi (2002), without going into the details, using the argument used for the Euler equations. It is easy to show, using the bounds in Proposition 6.2, that $\boldsymbol{B} \in C_{\mathrm{W}}([0,T_*];H^s(\mathbb{R}^n))$; that is, $\boldsymbol{B}$ is continuous in the weak topology of H^s. It thus suffices to show that $\|\boldsymbol{B}(\cdot)\|_{H^s}$ is continuous as a function of time. For fixed $\boldsymbol{u}$ such that $\nabla\boldsymbol{u} \in L^2(0,T_*;H^s(\mathbb{R}^n))$, proceeding analogously to Proposition 6.3, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}(t)\|_{H^s} \leq c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}^2,$$

and Gronwall's inequality shows that

$$\|\boldsymbol{B}(t)\|_{H^s} \leq \|\boldsymbol{B}_0\|_{H^s}\exp\left(\int_0^t \|\nabla\boldsymbol{u}(\tau)\|_{H^s}\,\mathrm{d}\tau\right),$$

and hence $\|\boldsymbol{B}(\cdot)\|_{H^s}$ is continuous from the right at time $t=0$; applying this bound to the equation started at an arbitrary time $\tau \in [0,T_*]$ shows that $\|\boldsymbol{B}(\cdot)\|_{H^s}$ is continuous from the right at time $t=\tau$. But the $\boldsymbol{B}$ equation is time-reversible, so $\|\boldsymbol{B}(\cdot)\|_{H^s}$ is continuous from the left at time $t=\tau$, and as τ was arbitrary $\|\boldsymbol{B}(\cdot)\|_{H^s}$ is continuous. This, combined with the fact that $\boldsymbol{B} \in C_{\mathrm{W}}([0,T_*];H^s(\mathbb{R}^n))$, yields that $\boldsymbol{B} \in C([0,T_*];H^s(\mathbb{R}^n))$. This completes the proof of Theorem 6.1.

6.2 Local Existence for Non-Resistive Stokes-MHD

Consider again the Stokes-MHD equations

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.6a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.6b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0 \tag{1.6c}$$

on the whole of $\mathbb{R}^n$, with divergence-free initial data $\boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$, for $s > n/2$.

In chapter 4 we established global existence (and uniqueness in 2D) of solutions of (1.2) with $\eta > 0$; equations (1.6) correspond to the case $\eta = 0$. In this section, we establish local existence and uniqueness of solutions for (1.6) (without magnetic diffusion) in H^s for $s > n/2$.

Theorem 6.7. *Let $n = 2,3$. For $s > n/2$, and initial data $\boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$ with $\nabla\cdot\boldsymbol{B}_0 = 0$, there exists a time $T_* = T_*(s,\|\boldsymbol{B}_0\|_{H^s}) > 0$ such that the equations (1.6) have a unique solution $(\boldsymbol{u},\boldsymbol{B})$, such that $\boldsymbol{B} \in C([0,T_*];H^s(\mathbb{R}^n))$ and $\boldsymbol{u} \in C([0,T_*];H^{s+1}(\mathbb{R}^n))$.*

In this case, we consider the truncated equations:

$$-\nu\Delta\boldsymbol{u}^R + \nabla p_*^R = (\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R, \tag{6.14a}$$

$$\frac{\partial \boldsymbol{B}^R}{\partial t} = \mathcal{S}_R[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{u}^R] - \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R], \tag{6.14b}$$

$$\nabla\cdot\boldsymbol{u}^R = \nabla\cdot\boldsymbol{B}^R = 0, \tag{6.14c}$$

with initial data $\boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$. Using standard elliptic regularity results in conjucntion with Lemma 6.5, we see that

$$\begin{aligned}\|\boldsymbol{u}^R - \boldsymbol{u}^{R'}\|_{H^{s+1}} &\le \frac{1}{\nu}\|(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R - (\boldsymbol{B}^{R'}\cdot\nabla)\boldsymbol{B}^{R'}\|_{H^{s-1}} \\ &\le \frac{1}{\nu}\left(\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{H^s}\|\boldsymbol{B}^R\|_{H^s} + \|\boldsymbol{B}^{R'}\|_{H^s}\|\boldsymbol{B}^R - \boldsymbol{B}^{R'}\|_{H^s}\right),\end{aligned} \tag{6.15}$$

so on V_R, $\boldsymbol{u}^R$ is a Lipschitz function of $\boldsymbol{B}^R$. Thus, as before, the second equation (for $\boldsymbol{B}$) is a Lipschitz ODE on the space V_R, and by Picard's theorem has a solution for as long as $\|\boldsymbol{B}^R\|_{H^s}$ remains finite.

By the same techniques as Proposition 6.2, we obtain the uniform bound

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}^R\|_{H^s}^2 + \nu\|\nabla\boldsymbol{u}^R\|_{H^s}^2 \le c\|\nabla\boldsymbol{u}^R\|_{H^s}\|\boldsymbol{B}^R\|_{H^s}^2,$$

and a Gronwall argument again shows there is some time T_* such that $\boldsymbol{B}^R$ are uniformly bounded in $L^\infty(0,T_*;H^s(\mathbb{R}^n))$. Furthermore, using Lemma 6.5, we obtain

$$\|\boldsymbol{u}\|_{H^{s+1}} \le \|(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\|_{H^{s-1}} \le \|\boldsymbol{B}\|_{H^s}^2,$$

so $\boldsymbol{u}^R$ are uniformly bounded in $L^\infty(0,T_*;H^{s+1}(\mathbb{R}^n))$.

An almost identical argument to Proposition 6.3 — which we omit here — shows that $\boldsymbol{B}^R \to \boldsymbol{B}$ strongly in $L^\infty(0,T_*;L^2(\mathbb{R}^n))$ and $\nabla\boldsymbol{u}^R \to \nabla\boldsymbol{u}$ strongly in $L^2(0,T_*;L^2(\mathbb{R}^n))$. Interpolation thus yields that, for any $s' < s$, $\boldsymbol{B}^R \to \boldsymbol{B}$ strongly in $L^\infty(0,T_*;H^{s'}(\mathbb{R}^n))$, and $\boldsymbol{u}^R \to \boldsymbol{u}$ strongly in $L^\infty(0,T_*;H^{s'+1}(\mathbb{R}^n))$. Hence $\Delta\boldsymbol{u}^R \to \Delta\boldsymbol{u}$ strongly in $L^\infty(0,T_*;H^{s'-1}(\mathbb{R}^n))$.

Convergence of the nonlinear terms is handled in the same way as the previous case, and thus $(\boldsymbol{u},\boldsymbol{B})$ solves (1.6) as an equality in $H^{s'-1}$. The Banach–Alaoglu theorem guarantees that the limit $\boldsymbol{u} \in L^\infty(0,T_*;H^{s+1}(\mathbb{R}^n))$ and $\boldsymbol{B} \in L^\infty(0,T_*;H^s(\mathbb{R}^n))$, and uniqueness is handled similarly to the previous case. Exactly the same argument as the previous case applies to show that in fact $\boldsymbol{B} \in C([0,T_*];H^s(\mathbb{R}^n))$; thence, an argument analogous to (6.15) for $\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)$ shows that $\boldsymbol{u} \in C([0,T_*];H^{s+1}(\mathbb{R}^n))$. This completes the proof of Theorem 6.7.

Chapter 7

Local Existence in Besov Spaces for Non-Resistive MHD

We consider again the non-resistive MHD equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta \boldsymbol{u} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \tag{1.5a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \tag{1.5b}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0 \tag{1.5c}$$

on the whole of $\mathbb{R}^n$ for $n = 2, 3$. In this chapter we work in Besov spaces, and we take divergence-free initial data $\boldsymbol{u}_0 \in B^{n/2-1}_{2,1}(\mathbb{R}^n)$ and $\boldsymbol{B}_0 \in B^{n/2}_{2,1}(\mathbb{R}^n)$. The space $B^{n/2}_{2,1}$ is the natural replacement for the space $H^{n/2}$: it is the largest Besov space which still embeds in L^∞ (unlike $H^{n/2}$). We prove the following theorem.

Theorem 7.1. *Let $n = 2, 3$. For $\boldsymbol{u}_0 \in B^{n/2-1}_{2,1}(\mathbb{R}^n)$ and $\boldsymbol{B}_0 \in B^{n/2}_{2,1}(\mathbb{R}^n)$ with $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$, there exists a time $T_* = T_*(\nu, \boldsymbol{u}_0, \|\boldsymbol{B}_0\|_{B^{n/2}_{2,1}}) > 0$ such that the equations* (1.5) *have at least one weak solution $(\boldsymbol{u}, \boldsymbol{B})$, with*

$$\boldsymbol{u} \in L^\infty([0, T_*]; B^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0, T_*; B^{n/2+1}_{2,1}(\mathbb{R}^n)),$$
$$\boldsymbol{B} \in L^\infty([0, T_*]; B^{n/2}_{2,1}(\mathbb{R}^n)).$$

The bulk of the work is in proving a priori estimates for an approximate version of the equations: two of the a priori estimates (in Section 7.1) apply equally in both 2D and 3D. However, the main estimate on the $\boldsymbol{u}$ equation involves the term $\int_0^T \|\boldsymbol{u}(t)\|^2_{H^{n/2}} \, \mathrm{d}t$ on the right-hand side: in 2D, this is easily taken care of using the energy inequality (see Section 7.2.1), but in 3D this needs a careful argument, based on the splitting method of Calderón (1990), to yield an $H^{1/2}$ estimate for Navier–Stokes (see Section 7.2.2).

The rest of the proof of Theorem 7.1 is outlined in Section 7.3. In addition we prove that, in 3D, the solution whose existence is asserted by Theorem 7.1 is unique.

Surprisingly, the proof of uniqueness in 2D is more difficult and remains open. Furthermore, note that we require the initial data to have finite energy, taking $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ in inhomogeneous Besov spaces rather than their homogeneous counterparts. For further discussion on both these issues, see the conclusion (Chapter 8).

7.1 A Priori Estimates

We first prove the two main a priori estimates that we will use in the existence proof: to streamline the presentation we prove the estimates formally for $\boldsymbol{u}$ and $\boldsymbol{B}$ which solve equations (1.5).

Proposition 7.2. *If* $(\boldsymbol{u}, \boldsymbol{B})$ *solve equations* (1.5) *on* $[0,T]$*, then there is a constant* c_1 *such that, for all* $t \in [0,T]$*,*

$$\|\boldsymbol{B}(t)\|_{\dot{B}^{n/2}_{2,1}} \leq \|\boldsymbol{B}_0\|_{\dot{B}^{n/2}_{2,1}} \exp\left(c_1 \int_0^t \|\nabla \boldsymbol{u}(s)\|_{\dot{B}^{n/2}_{2,1}} \, \mathrm{d}s\right).$$

Before embarking on the proof, we state a lemma we require, which is a particular case of Lemma 2.100 from Bahouri et al. (2011).

Lemma 7.3. *Let* $-1-n/2 < \sigma < 1+n/2$ *and* $1 \leq r \leq \infty$*. Let* $\boldsymbol{v}$ *be a divergence-free vector field on* $\mathbb{R}^n$*, and set* $Q_j := [(\boldsymbol{v}\cdot\nabla), \dot{\triangle}_j]f$*. There exists a constant* $C = C(\sigma, n)$*, such that*

$$\left\|\left(2^{j\sigma}\|Q_j\|_{L^2}\right)_j\right\|_{\ell^r} \leq C\|\nabla \boldsymbol{v}\|_{\dot{B}^{n/2}_{2,\infty}\cap L^\infty}\|f\|_{\dot{B}^{\sigma}_{2,r}}.$$

Proof of Proposition 7.2. Given $j \in \mathbb{Z}$, apply the homogeneous Littlewood–Paley operator $\dot{\triangle}_j$ (see Section 2.4.1) to the equation (1.5b) for $\boldsymbol{B}$ to obtain

$$\frac{\partial}{\partial t}\dot{\triangle}_j \boldsymbol{B} + \dot{\triangle}_j[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] = \dot{\triangle}_j[(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}].$$

As $\dot{B}^{n/2}_{2,1}$ is an algebra (see equation (2.22)), we have

$$\|(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}} \leq \|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}}\|\nabla \boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}.$$

By the standard trick (2.21), we may write

$$\|\dot{\triangle}_j[(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}]\|_{L^2} \leq C d_j(t) 2^{-jn/2}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}}\|\nabla \boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}$$

where $d_j(t)$ denotes a sequence in $\ell^1(\mathbb{Z})$ whose sum is 1.

For the term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$, we use Bony's paraproduct decomposition:

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}_\ell=\sum_{k=1}^{n}[\dot{T}_{\boldsymbol{u}_k}\partial_k\boldsymbol{B}_\ell+\dot{T}_{\partial_k\boldsymbol{B}_\ell}\boldsymbol{u}_k+\dot{R}(\boldsymbol{u}_k,\partial_k\boldsymbol{B}_\ell)].$$

Consider the second term $\dot{T}_{\partial_k\boldsymbol{B}_\ell}\boldsymbol{u}_k$: by Lemma 2.11 we have

$$\begin{aligned}\|\dot{T}_{\partial_k\boldsymbol{B}_\ell}\boldsymbol{u}_k\|_{\dot{B}^{n/2}_{2,1}}&\le c\sum_{k=1}^{n}\|\partial_k\boldsymbol{B}_\ell\|_{\dot{B}^{-1}_{\infty,\infty}}\|\boldsymbol{u}_k\|_{\dot{B}^{n/2+1}_{2,1}}\\&\le c\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}}\|\nabla\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}},\end{aligned}$$

where we have used that $\dot{B}^{n/2}_{2,1}\hookrightarrow\dot{B}^{0}_{\infty,\infty}$ (by Proposition 2.9). For the third term $\dot{R}(\boldsymbol{u}_k,\partial_k\boldsymbol{B}_\ell)$, we apply Lemma 2.12:

$$\begin{aligned}\|\dot{R}(\boldsymbol{u}_k,\partial_k\boldsymbol{B}_\ell)\|_{\dot{B}^{n/2}_{2,1}}&\le c\sum_{k=1}^{n}\|\boldsymbol{u}_k\|_{\dot{B}^{n/2+1}_{2,1}}\|\partial_k\boldsymbol{B}_\ell\|_{\dot{B}^{-1}_{\infty,\infty}}\\&\le c\|\nabla\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}},\end{aligned}$$

as above. Using the standard trick (2.21), we obtain

$$\begin{aligned}\sum_{k=1}^{n}\|\dot{\triangle}_j\dot{T}_{\partial_k\boldsymbol{B}_\ell}\boldsymbol{u}_k\|_{L^2}&\le cd_j(t)2^{-jn/2}\|\nabla\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}},\\\sum_{k=1}^{n}\|\dot{\triangle}_j\dot{R}(\boldsymbol{u}_k,\partial_k\boldsymbol{B}_\ell)\|_{L^2}&\le cd_j(t)2^{-jn/2}\|\nabla\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}}.\end{aligned}$$

For the term $\dot{T}_{\boldsymbol{u}_k}\partial_k\boldsymbol{B}_\ell$, let us write

$$\begin{aligned}\sum_{k=1}^{n}\dot{\triangle}_j\dot{T}_{\boldsymbol{u}_k}\partial_k\boldsymbol{B}_\ell&=\sum_{j'\in\mathbb{Z}}\sum_{k=1}^{n}\dot{\triangle}_j\left(\dot{S}_{j'-1}\boldsymbol{u}_k\partial_k\dot{\triangle}_{j'}\boldsymbol{B}_\ell\right)\\&=\sum_{k=1}^{n}\dot{S}_{j-1}\boldsymbol{u}_k\partial_k\dot{\triangle}_j\boldsymbol{B}_\ell+\sum_{j'\in\mathbb{Z}}\sum_{k=1}^{n}(\dot{S}_{j'-1}\boldsymbol{u}_k-\dot{S}_{j-1}\boldsymbol{u}_k)\partial_k\dot{\triangle}_j\dot{\triangle}_{j'}\boldsymbol{B}_\ell\\&\quad+\sum_{j'\in\mathbb{Z}}\sum_{k=1}^{n}[\dot{\triangle}_j,\dot{S}_{j'-1}\boldsymbol{u}_k\partial_k]\left(\dot{\triangle}_{j'}\boldsymbol{B}_\ell\right)\\&=:(\dot{S}_{j-1}\boldsymbol{u}\cdot\nabla)\dot{\triangle}_j\boldsymbol{B}_\ell+P_j+Q_j.\end{aligned}$$

For P_j, by (2.20c) we have

$$P_j := \sum_{|j-j'|\le 1}\sum_{k=1}^{n}(\dot S_{j'-1}\boldsymbol{u}_k - \dot S_{j-1}\boldsymbol{u}_k)\dot\triangle_j\dot\triangle_{j'}\partial_k\boldsymbol{B}_\ell$$
$$= \sum_{k=1}^{n}(\dot\triangle_{j-1}\boldsymbol{u}_k)(\dot\triangle_j\dot\triangle_{j+1}\partial_k\boldsymbol{B}_\ell) - \sum_{k=1}^{n}(\dot\triangle_{j-2}\boldsymbol{u}_k)(\dot\triangle_j\dot\triangle_{j-1}\partial_k\boldsymbol{B}_\ell),$$

so as $\|\dot\triangle_j\partial_k\boldsymbol{B}\|_{L^2}\simeq 2^j\|\dot\triangle_j\boldsymbol{B}\|_{L^2}$ we have

$$\begin{aligned}2^{jn/2}\|P_j\|_{L^2} &\le c\Big(4\cdot 2^{j-1}\|\dot\triangle_{j-1}\boldsymbol{u}\|_{L^\infty}2^{jn/2}\|\dot\triangle_j\boldsymbol{B}_\ell\|_{L^2}\\ &\qquad + 2\cdot 2^{j-2}\|\dot\triangle_{j-2}\boldsymbol{u}\|_{L^\infty}2^{jn/2}\|\dot\triangle_j\boldsymbol{B}_\ell\|_{L^2}\Big)\\ &\le c d_j(t)\|\boldsymbol{u}\|_{\dot B^1_{\infty,\infty}}\|\boldsymbol{B}\|_{\dot B^{n/2}_{2,1}}\\ &\le c d_j(t)\|\nabla\boldsymbol{u}\|_{\dot B^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot B^{n/2}_{2,1}}.\end{aligned}$$

For Q_j, we apply Lemma 7.3: note that

$$Q_j := \sum_{j'\in\mathbb{Z}}[\dot\triangle_j, \dot S_{j'-1}(\boldsymbol{u}\cdot\nabla)]\left(\dot\triangle_{j'}\boldsymbol{B}_\ell\right)$$

so

$$\begin{aligned}\left\|\left(2^{jn/2}\|Q_j\|_{L^2}\right)_j\right\|_{\ell^1} &\le c\|\nabla\boldsymbol{u}\|_{\dot B^{n/2}_{2,\infty}\cap L^\infty}\|\boldsymbol{B}\|_{\dot B^{n/2}_{2,1}}\\ &\le c\|\nabla\boldsymbol{u}\|_{\dot B^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot B^{n/2}_{2,1}}\end{aligned}$$

since $\dot B^{n/2}_{2,1}$ embeds continuously in both $\dot B^{n/2}_{2,\infty}$ (by Proposition 2.9) and L^∞ (by Proposition 2.10). So by the standard trick

$$\|Q_j\|_{L^2}\le c d_j(t)2^{-jn/2}\|\nabla\boldsymbol{u}\|_{\dot B^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot B^{n/2}_{2,1}}.$$

By combining all the above estimates, we obtain

$$\frac{\partial}{\partial t}\dot\triangle_j\boldsymbol{B} + (\dot S_{j-1}\boldsymbol{u}\cdot\nabla)\dot\triangle_j\boldsymbol{B} = F_j(t), \tag{7.2}$$

where

$$\|F_j(t)\|_{L^2}\le c d_j(t)2^{-jn/2}\|\nabla\boldsymbol{u}\|_{\dot B^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot B^{n/2}_{2,1}}.$$

Taking the inner product of (7.2) with $\dot\triangle_j\boldsymbol{B}$ and using the fact that $\boldsymbol{u}$ (and hence

$\dot{S}_{j-1}\boldsymbol{u}$) is divergence-free, we obtain

$$2^{jn/2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\triangle}_j\boldsymbol{B}\|_{L^2} \le 2cd_j(t)\|\nabla\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}}$$

so summing in j yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}} \le c\|\nabla\boldsymbol{u}\|_{\dot{B}^{n/2}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{n/2}_{2,1}}$$

and the result follows by Gronwall's inequality. □

Our second estimate, for the $\boldsymbol{u}$ equation alone, is stated for a generic forcing $\boldsymbol{f}$.

Proposition 7.4. *Let $\boldsymbol{f} \in L^1(0,T;\dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n))$. Suppose $\boldsymbol{u}$ solves*

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{7.3a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{7.3b}$$

on the time interval $[0,T]$. Then there is a constant c_2 such that, for all $t \in [0,T]$,

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{\dot{B}^{n/2-1}_{2,1}} &+ \nu\int_0^t \|\nabla\boldsymbol{u}(s)\|_{\dot{B}^{n/2}_{2,1}}\,\mathrm{d}s \\ &\le \|\boldsymbol{u}_0\|_{\dot{B}^{n/2-1}_{2,1}} + c_2\int_0^t \|\boldsymbol{u}(s)\|^2_{H^{n/2}}\,\mathrm{d}s + c_2\int_0^t \|\boldsymbol{f}(s)\|_{\dot{B}^{n/2-1}_{2,1}}\,\mathrm{d}s.\end{aligned}$$

Note that in the particular case $\boldsymbol{f} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = \nabla\cdot(\boldsymbol{B}\otimes\boldsymbol{B})$, we have

$$\|\boldsymbol{f}\|_{\dot{B}^{n/2-1}_{2,1}} = \|\nabla\cdot(\boldsymbol{B}\otimes\boldsymbol{B})\|_{\dot{B}^{n/2-1}_{2,1}} \le \|\boldsymbol{B}\|^2_{\dot{B}^{n/2}_{2,1}} \tag{7.4}$$

since $\dot{B}^{n/2}_{2,1}$ is an algebra.

For the proof we will need the following lemma.

Lemma 7.5 (Lemma 1.1 from Chemin (1992))**.** *Let $\boldsymbol{v}$ be a divergence-free vector field, and let s, r, r', r'' be four real numbers such that $r + r' + r'' = n/2 + 1 + 2s$, $r + r' > 0$, $0 \le r < n/2 + 1$ and $r' < n/2 + 1$. Then there exists a constant C such that*

$$\langle\Lambda^s[(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}],\Lambda^s\boldsymbol{w}\rangle \le C\left(\|\boldsymbol{v}\|_{H^r}\|\boldsymbol{w}\|_{\dot{H}^{r'}} + \|\boldsymbol{w}\|_{H^r}\|\boldsymbol{v}\|_{\dot{H}^{r'}}\right)\|\boldsymbol{w}\|_{\dot{H}^{r''}}.$$

In particular, taking $s = n/2-1$, $r = r' = n/2$ and $r'' = n/2 - 1$ yields

$$\langle\Lambda^{n/2-1}[(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}],\Lambda^{n/2-1}\boldsymbol{w}\rangle \le C\left(\|\boldsymbol{v}\|_{H^{n/2}}\|\boldsymbol{w}\|_{\dot{H}^{n/2}} + \|\boldsymbol{w}\|_{H^{n/2}}\|\boldsymbol{v}\|_{\dot{H}^{n/2}}\right)\|\boldsymbol{w}\|_{\dot{H}^{n/2-1}}. \tag{7.5}$$

Proof of Proposition 7.4. Applying the Littlewood–Paley operator $\dot{\triangle}_j$ to equation (7.3) yields

$$\frac{\partial}{\partial t}\dot{\triangle}_j \boldsymbol{u} + \dot{\triangle}_j[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] - \nu\Delta\dot{\triangle}_j\boldsymbol{u} + \nabla\dot{\triangle}_j p = \dot{\triangle}_j \boldsymbol{f}.$$

Taking the inner product with $\dot{\triangle}_j \boldsymbol{u}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\triangle}_j\boldsymbol{u}\|_{L^2}^2 + c\nu 2^{2j}\|\dot{\triangle}_j\boldsymbol{u}\|_{L^2}^2 \le \left|\langle\dot{\triangle}_j[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}],\dot{\triangle}_j\boldsymbol{u}\rangle\right| + \left|\langle\dot{\triangle}_j\boldsymbol{f},\dot{\triangle}_j\boldsymbol{u}\rangle\right|$$

Applying the estimate from equation (7.5) yields

$$\begin{aligned}\langle\Lambda^{n/2-1}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}],\Lambda^{n/2-1}\boldsymbol{u}\rangle &\le c\|\boldsymbol{u}\|_{H^{n/2}}\|\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\boldsymbol{u}\|_{\dot{H}^{n/2-1}}\\ &\le c\|\boldsymbol{u}\|_{H^{n/2}}^2\|\boldsymbol{u}\|_{\dot{B}^{n/2-1}_{2,1}}.\end{aligned}$$

Decomposing each term on the left-hand side, we obtain

$$\sum_{j,j'\in\mathbb{Z}}\langle 2^{j(n/2-1)}\dot{\triangle}_j[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}],2^{j'(n/2-1)}\dot{\triangle}_{j'}\boldsymbol{u}\rangle \le c\|\boldsymbol{u}\|_{H^{n/2}}^2\|\boldsymbol{u}\|_{\dot{B}^{n/2-1}_{2,1}}$$

Just taking the sum of the "diagonal" terms yields

$$\sum_{j\in\mathbb{Z}}2^{j(n-2)}\langle\dot{\triangle}_j[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}],\dot{\triangle}_j\boldsymbol{u}\rangle \le c\|\boldsymbol{u}\|_{H^{n/2}}^2\|\boldsymbol{u}\|_{\dot{B}^{n/2-1}_{2,1}}.$$

By the standard trick (and dividing by $2^{j(n-2)}$) we obtain

$$\left|\langle\dot{\triangle}_j[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}],\dot{\triangle}_j\boldsymbol{u}\rangle\right| \le c d_j(t)2^{-j(n/2-1)}\|\boldsymbol{u}\|_{H^{n/2}}^2\|\dot{\triangle}_j\boldsymbol{u}\|_{L^2}.$$

Hence

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\triangle}_j\boldsymbol{u}(t)\|_{L^2}^2 + c\nu 2^{2j}\|\dot{\triangle}_j\boldsymbol{u}(t)\|_{L^2}^2\\ &\qquad\le c d_j(t)2^{-j(n/2-1)}\left(\|\boldsymbol{u}(t)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(t)\|_{\dot{B}^{n/2-1}_{2,1}}\right)\|\dot{\triangle}_j\boldsymbol{u}(t)\|_{L^2}.\end{aligned}$$

Dividing through by $\|\dot{\triangle}_j\boldsymbol{u}(t)\|_{L^2}$ and multiplying by $e^{c\nu 2^{2j}t}$ yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{c\nu 2^{2j}t}\|\dot{\triangle}_j\boldsymbol{u}(t)\|_{L^2}\right)\\ &\qquad\le c e^{c\nu 2^{2j}t}d_j(t)2^{-j(n/2-1)}\left(\|\boldsymbol{u}(t)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(t)\|_{\dot{B}^{n/2-1}_{2,1}}\right).\end{aligned}$$

Integrating in time from 0 to t yields

$$
\begin{aligned}
\|\dot{\triangle}_j \boldsymbol{u}(t)\|_{L^2} &\le \|\dot{\triangle}_j \boldsymbol{u}_0\|_{L^2} e^{-c\nu 2^{2j} t} \qquad (7.6)\\
&\quad + c2^{-j(n/2-1)} \int_0^t d_j(s) e^{-c\nu 2^{2j}(t-s)} \left(\|\boldsymbol{u}(s)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(s)\|_{\dot{B}^{n/2-1}_{2,1}} \right) \mathrm{d}s.
\end{aligned}
$$

As $e^{-c\nu 2^{2j}t} \le 1$ for all t, multiplying (7.6) by $2^{j(n/2-1)}$ and summing in j yields

$$
\|\boldsymbol{u}(t)\|_{\dot{B}^{n/2-1}_{2,1}} \le \|\boldsymbol{u}_0\|_{\dot{B}^{n/2-1}_{2,1}} + c \int_0^t \left(\|\boldsymbol{u}(s)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(s)\|_{\dot{B}^{n/2-1}_{2,1}} \right) \mathrm{d}s.
$$

Taking the L^∞ norm over $t \in [0,T]$ yields

$$
\|\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{n/2-1}_{2,1})} \le \|\boldsymbol{u}_0\|_{\dot{B}^{n/2-1}_{2,1}} + c \int_0^T \left(\|\boldsymbol{u}(t)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(t)\|_{\dot{B}^{n/2-1}_{2,1}} \right) \mathrm{d}t.
$$

Multiplying (7.6) by $\nu 2^{j(n/2+1)}$ and then taking the L^1 norm over $t \in [0,T]$ yields

$$
\begin{aligned}
\nu 2^{j(n/2)} \|\dot{\triangle}_j \nabla \boldsymbol{u}\|_{L^1(0,T;L^2)} &\le 2^{j(n/2-1)} \|\dot{\triangle}_j \boldsymbol{u}_0\|_{L^2} \int_0^T \nu 2^{2j} e^{-c\nu 2^{2j}t} \,\mathrm{d}t\\
&\quad + c \int_0^T \int_0^t d_j(s) \nu 2^{2j} e^{-c\nu 2^{2j}(t-s)} \left(\|\boldsymbol{u}(s)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(s)\|_{\dot{B}^{n/2-1}_{2,1}} \right) \mathrm{d}s \,\mathrm{d}t.
\end{aligned}
$$

Using Young's inequality for convolutions and the fact that

$$
\int_0^T c\nu 2^{2j} e^{-c\nu 2^{2j}t} \,\mathrm{d}t = 1 - e^{-c\nu 2^{2j}T} \le 1
$$

yields

$$
\begin{aligned}
\nu 2^{j(n/2)} \|\dot{\triangle}_j \nabla \boldsymbol{u}\|_{L^1(0,T;L^2)} &\le c2^{j(n/2-1)} \|\dot{\triangle}_j \boldsymbol{u}_0\|_{L^2}\\
&\quad + c \int_0^T d_j(t) \left(\|\boldsymbol{u}(t)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(t)\|_{\dot{B}^{n/2-1}_{2,1}} \right) \mathrm{d}t.
\end{aligned}
$$

Summation in j and the Monotone Convergence Theorem yields

$$
\nu \|\nabla \boldsymbol{u}\|_{L^1(0,T;\dot{B}^{n/2}_{2,1})} \le \|\boldsymbol{u}_0\|_{\dot{B}^{n/2-1}_{2,1}} + c \int_0^T \left(\|\boldsymbol{u}(t)\|_{H^{n/2}}^2 + \|\boldsymbol{f}(t)\|_{\dot{B}^{n/2-1}_{2,1}} \right) \mathrm{d}t.
$$

This completes the proof. $\square$

7.2 Uniform Bounds in 2D and 3D

To turn our a priori estimates into a rigorous proof, let us consider again the truncated MHD equations from Chapter 6:

$$\frac{\partial \boldsymbol{u}^R}{\partial t} - \nu\Delta\boldsymbol{u}^R + \nabla p_*^R = \mathcal{S}_R[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R] - \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R], \tag{7.7a}$$

$$\frac{\partial \boldsymbol{B}^R}{\partial t} = \mathcal{S}_R[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{u}^R] - \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R], \tag{7.7b}$$

$$\nabla\cdot\boldsymbol{u}^R = \nabla\cdot\boldsymbol{B}^R = 0, \tag{7.7c}$$

with initial data $\boldsymbol{u}^R(0) = \mathcal{S}_R\boldsymbol{u}_0$, $\boldsymbol{B}^R(0) = \mathcal{S}_R\boldsymbol{B}_0$.

Repeating the a priori estimates from Proposition 7.2 we obtain

$$\|\boldsymbol{B}^R(t)\|_{\dot{B}^{n/2}_{2,1}} \le \|\boldsymbol{B}_0\|_{\dot{B}^{n/2}_{2,1}} \exp\left(c_1\int_0^t \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^{n/2}_{2,1}}\,\mathrm{d}s\right),$$

where the constant c_1 is independent of R. Repeating Proposition 7.4 for the equation

$$\frac{\partial \boldsymbol{u}^R}{\partial t} + \mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R] - \nu\Delta\boldsymbol{u}^R + \nabla p_*^R = \boldsymbol{f}^R, \tag{7.8a}$$

$$\nabla\cdot\boldsymbol{u}^R = 0. \tag{7.8b}$$

yields

$$\begin{aligned}\|\boldsymbol{u}^R(t)\|_{\dot{B}^{n/2-1}_{2,1}} &+ \nu\int_0^t \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^{n/2}_{2,1}}\,\mathrm{d}s \\ &\le \|\boldsymbol{u}_0\|_{\dot{B}^{n/2-1}_{2,1}} + c_2\int_0^t \|\boldsymbol{u}^R(s)\|^2_{H^{n/2}}\,\mathrm{d}s + c_2\int_0^t \|\boldsymbol{f}^R(s)\|_{\dot{B}^{n/2-1}_{2,1}}\,\mathrm{d}s,\end{aligned}$$

where the constant c_2 is independent of R.

Turning these estimates into uniform bounds on $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ which are independent of R depends on the dimension, so we consider the 2D and 3D cases separately. However, in both cases we will make use of the following standard energy estimate:

$$\sup_{t\in[0,T]}\|\boldsymbol{u}^R(t)\|^2_{L^2} + \sup_{t\in[0,T]}\|\boldsymbol{B}^R(t)\|^2_{L^2} + \nu\int_0^T \|\nabla\boldsymbol{u}^R(s)\|^2_{L^2}\,\mathrm{d}s \le 2(\|\boldsymbol{u}_0\|^2_{L^2} + \|\boldsymbol{B}_0\|^2_{L^2}) \tag{7.9}$$

for any $T > 0$, which can be obtained by taking the inner product of (1.5a) with $\boldsymbol{u}^R$, the inner product of (1.5b) with $\boldsymbol{B}^R$, and adding (see also equation (6.5)).

7.2.1 Uniform Bounds in Two Dimensions

In 2D, the term $\int_0^t \|\boldsymbol{u}^R(s)\|^2_{H^{n/2}}\,\mathrm{d}s$ is simply $\int_0^t \|\boldsymbol{u}^R(s)\|^2_{H^1}\,\mathrm{d}s$. Using the standard energy estimate (7.9) we may bound this as follows:

$$\begin{aligned}\int_0^t \|\boldsymbol{u}^R(s)\|^2_{H^1}\,\mathrm{d}s &\le \int_0^t \|\boldsymbol{u}^R(s)\|^2_{L^2}\,\mathrm{d}s + \int_0^t \|\nabla\boldsymbol{u}^R(s)\|^2_{L^2}\,\mathrm{d}s \\ &\le 2\left(t+\frac{1}{\nu}\right)(\|\boldsymbol{u}_0\|^2_{L^2}+\|\boldsymbol{B}_0\|^2_{L^2}).\end{aligned} \tag{7.10}$$

Using this, we show that $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ are uniformly bounded.

Theorem 7.6. *There is a time $T_* = T_*(\nu, \|\boldsymbol{u}_0\|_{B^0_{2,1}}, \|\boldsymbol{B}_0\|_{B^1_{2,1}}) > 0$ such that*

$$\begin{aligned}&\boldsymbol{u}^R \textit{ is uniformly bounded in } L^\infty(0,T_*;\dot{B}^0_{2,1}(\mathbb{R}^2))\cap L^1(0,T_*;\dot{B}^2_{2,1}(\mathbb{R}^2)),\\ &\boldsymbol{B}^R \textit{ is uniformly bounded in } L^\infty(0,T_*;\dot{B}^1_{2,1}(\mathbb{R}^2)).\end{aligned}$$

Proof. Let

$$\begin{aligned}M_1 &= \|\boldsymbol{u}_0\|_{\dot{B}^0_{2,1}} + \frac{2c_2}{\nu}(\|\boldsymbol{u}_0\|^2_{L^2}+\|\boldsymbol{B}_0\|^2_{L^2}),\\ M_2 &= 2c_2(\|\boldsymbol{u}_0\|^2_{L^2}+\|\boldsymbol{B}_0\|^2_{L^2}).\end{aligned}$$

Substituting from equation (7.10) into Proposition 7.4, we obtain

$$\|\boldsymbol{u}^R(t)\|_{\dot{B}^0_{2,1}} + \nu\int_0^t \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^1_{2,1}}\,\mathrm{d}s \le M_1 + M_2 t + c_2\int_0^t \|\boldsymbol{f}^R(s)\|_{\dot{B}^0_{2,1}}\,\mathrm{d}s.$$

Using (7.4) and substituting in from Proposition 7.2, we obtain

$$\begin{aligned}&\|\boldsymbol{u}^R(t)\|_{\dot{B}^0_{2,1}} + \nu\int_0^t \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^1_{2,1}}\,\mathrm{d}s\\ &\qquad\le M_1 + M_2 t + \int_0^t \|\boldsymbol{B}_0\|^2_{\dot{B}^1_{2,1}} c_2\exp\left(2c_1\int_0^\tau \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^1_{2,1}}\,\mathrm{d}s\right)\mathrm{d}\tau.\\ &\qquad\le M_1 + M_2 t + M_3 t\exp\left(2c_1\int_0^t \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^1_{2,1}}\,\mathrm{d}s\right),\end{aligned}$$

where $M_3 = c_1\|\boldsymbol{B}_0\|^2_{\dot{B}^1_{2,1}}$. Let

$$\begin{aligned}X_R(t) &= \|\boldsymbol{u}^R(t)\|_{\dot{B}^0_{2,1}},\\ Y_R(t) &= \nu\int_0^t \|\nabla\boldsymbol{u}^R(s)\|_{\dot{B}^1_{2,1}}\,\mathrm{d}s.\end{aligned}$$

Then we can rewrite the last inequality as

$$X_R(t) + Y_R(t) \le M_1 + M_2 t + M_3 t \exp(2c_1 Y_R(t)/\nu). \tag{7.11}$$

Set

$$T_* = \min\left\{ \frac{M_1}{M_2}, \frac{M_1}{M_3} \exp(-6c_1 M_1/\nu) \right\}.$$

It remains to show that $X_R(t) + Y_R(t) \le 3M_1$ for all $t \in [0, T_*]$ and all $R > 0$. To that end, note that $Y_R(t)$ is continuous and $Y_R(0) = 0$. Now, suppose $t < T_*$ and $Y_R(t) \le 3M_1$; then

$$\begin{aligned} Y_R(t) &\le M_1 + M_2 t + M_3 t \exp(2c_1 Y_R(t)/\nu) \\ &< M_1 + M_1 + M_1 \exp\left(\frac{2c_1}{\nu} [Y_R(t) - 3M_1] \right) \\ &\le 3M_1. \end{aligned}$$

This means that $Y_R(t)$ can never equal $3M_1$ on the interval $[0, T_*)$; so $Y_R(t) < 3M_1$ for all $t \in [0, T_*)$. The result follows from inequality (7.11) and Proposition 7.2. $\square$

Before moving onto the 3D case, it is worth noting that in 2D the existence time T_* depends only on the norm $\|\boldsymbol{u}_0\|_{B^0_{2,1}}$ rather than the whole of $\boldsymbol{u}_0$.

7.2.2 Uniform Bounds in Three Dimensions

In 3D, we take initial data $\boldsymbol{u}_0 \in B^{1/2}_{2,1}(\mathbb{R}^3)$ and $\boldsymbol{B}_0 \in B^{3/2}_{2,1}(\mathbb{R}^3)$. Instead of being able to use the energy inequality, we require the following auxiliary estimate to bound $\int_0^t \|\nabla \boldsymbol{u}^R(s)\|^2_{\dot{H}^{1/2}}\,\mathrm{d}s$.

Proposition 7.7. *There exist constants c_3 and c_4 and a time $T_1 = T_1(\nu, \boldsymbol{u}_0)$ such that, if $T \le T_1$, $R > 0$ and*

$$\int_0^T \|\boldsymbol{B}^R(s)\|^2_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s \le \frac{\nu}{2(c_3 c_4)^{1/4}} =: C_*, \tag{7.12}$$

the solution $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ of (7.7) satisfies

$$\begin{aligned} \int_0^T \|\nabla \boldsymbol{u}^R(s)\|^2_{\dot{H}^{1/2}}\,\mathrm{d}s &\le \frac{1}{\nu}\|\boldsymbol{u}_0\|^2_{\dot{H}^{1/2}} + \frac{8c_3}{\nu^3}\|\boldsymbol{u}_0\|^4_{\dot{H}^{1/2}} \\ &\quad + \frac{3}{2\nu}\left(\int_0^T \|\boldsymbol{B}^R(s)\|^2_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s \right)^2 + \frac{4c_3}{\nu^3}\left(\int_0^T \|\boldsymbol{B}^R(s)\|^2_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s \right)^4. \end{aligned} \tag{7.13}$$

Note carefully that the estimate (7.13) is conditional on assumption (7.12) holding: once we have proved the proposition, we will require a further lemma to ensure that there is a time such that assumption (7.12) holds, and thus avoid a circular argument.

Proof of Proposition 7.7. The proof is based on the proof of Theorem 1 in Marín-Rubio, Robinson & Sadowski (2013), which in turn is based on the proof of Theorem 3.4 in Chemin et al. (2006); the original idea of splitting the equation is due to Calderón (1990).

First, let us consider the Stokes equation with initial data $\boldsymbol{u}_0$:

$$\frac{\partial \boldsymbol{h}}{\partial t} - \nu \Delta \boldsymbol{h} + \nabla p_{\boldsymbol{h}} = 0, \tag{7.14a}$$

$$\nabla \cdot \boldsymbol{h} = 0, \tag{7.14b}$$

$$\boldsymbol{h}(0) = \boldsymbol{u}_0, \tag{7.14c}$$

Thanks to the properties of the Stokes equation and of Fourier truncations, the solution of the equation

$$\frac{\partial \boldsymbol{h}^R}{\partial t} - \nu \Delta \boldsymbol{h}^R + \nabla p_{\boldsymbol{h}}^R = 0, \tag{7.15a}$$

$$\nabla \cdot \boldsymbol{h}^R = 0, \tag{7.15b}$$

$$\boldsymbol{h}^R(0) = \mathcal{S}_R \boldsymbol{u}_0, \tag{7.15c}$$

is given by $\boldsymbol{h}^R = \mathcal{S}_R \boldsymbol{h}$.

Let us decompose $\boldsymbol{u}^R = \boldsymbol{h}^R + \boldsymbol{v}^R + \boldsymbol{w}^R$, where $\boldsymbol{v}^R$ and $\boldsymbol{w}^R$ satisfy

$$\frac{\partial \boldsymbol{v}^R}{\partial t} - \nu \Delta \boldsymbol{v}^R + \nabla p_{\boldsymbol{v}}^R = \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{B}^R], \tag{7.16a}$$

$$\nabla \cdot \boldsymbol{v}^R = 0, \tag{7.16b}$$

$$\boldsymbol{v}^R(0) = 0, \tag{7.16c}$$

and

$$\frac{\partial \boldsymbol{w}^R}{\partial t} - \nu \Delta \boldsymbol{w}^R + \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{u}^R] + \nabla p_{\boldsymbol{w}}^R = 0, \tag{7.17a}$$

$$\nabla \cdot \boldsymbol{w}^R = 0, \tag{7.17b}$$

$$\boldsymbol{w}^R(0) = 0, \tag{7.17c}$$

respectively.

Applying $\Lambda^{1/2}$ to (7.17a) and taking the inner product with $\Lambda^{1/2}\boldsymbol{w}^R$ yields

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}^R\|^2_{\dot{H}^{1/2}} + \nu\|\boldsymbol{w}^R\|^2_{\dot{H}^{3/2}} &= \langle \Lambda^{1/2}\mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R], \Lambda^{1/2}\boldsymbol{w}^R\rangle \\
&= \langle (\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R, \Lambda\boldsymbol{w}^R\rangle \\
&\leq \|\boldsymbol{u}^R\|_{L^6}\|\nabla\boldsymbol{u}^R\|_{L^2}\|\Lambda\boldsymbol{w}^R\|_{L^3} \\
&\leq c\|\boldsymbol{u}^R\|^2_{\dot{H}^1}\|\boldsymbol{w}^R\|_{\dot{H}^{3/2}} \\
&\leq c\left(\|\boldsymbol{h}^R\|^2_{\dot{H}^1} + \|\boldsymbol{v}^R\|^2_{\dot{H}^1} + \|\boldsymbol{w}^R\|^2_{\dot{H}^1}\right)\|\boldsymbol{w}^R\|_{\dot{H}^{3/2}} \\
&\leq c\|\boldsymbol{h}^R\|^2_{\dot{H}^1}\|\boldsymbol{w}^R\|_{\dot{H}^{3/2}} + c\|\boldsymbol{v}^R\|^2_{\dot{H}^1}\|\boldsymbol{w}^R\|_{\dot{H}^{3/2}} \\
&\qquad + c\|\boldsymbol{w}^R\|_{\dot{H}^{1/2}}\|\boldsymbol{w}^R\|^2_{\dot{H}^{3/2}}
\end{aligned}$$

by interpolation. Using Young's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}^R\|^2_{\dot{H}^{1/2}} + \nu\|\boldsymbol{w}^R\|^2_{\dot{H}^{3/2}} \leq \frac{c_3}{\nu}\|\boldsymbol{h}^R\|^4_{\dot{H}^1} + \frac{c_3}{\nu}\|\boldsymbol{v}^R\|^4_{\dot{H}^1} + c_4\|\boldsymbol{w}^R\|_{\dot{H}^{1/2}}\|\boldsymbol{w}\|^2_{\dot{H}^{3/2}}.$$

For any $T > t > 0$, integrating in time over $[0, t]$ yields

$$\begin{aligned}
&\|\boldsymbol{w}^R(t)\|^2_{\dot{H}^{1/2}} + \nu\int_0^t \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s \\
&\quad\leq \frac{c_3}{\nu}\int_0^t \|\boldsymbol{h}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s + \frac{c_3}{\nu}\int_0^t \|\boldsymbol{v}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s \\
&\qquad\qquad + c_4\int_0^t \|\boldsymbol{w}^R(s)\|_{\dot{H}^{1/2}}\|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s \\
&\quad\leq \frac{c_3}{\nu}\int_0^T \|\boldsymbol{h}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s + \frac{c_3}{\nu}\int_0^T \|\boldsymbol{v}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s \\
&\qquad\qquad + \frac{1}{2}\sup_{s\in[0,T]}\|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{1/2}} + \frac{c_4}{2}\left(\int_0^T \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s\right)^2,
\end{aligned}$$

so taking the supremum on the left-hand side over $t \in [0, T]$ yields

$$\begin{aligned}
&\sup_{s\in[0,T]}\|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{1/2}} + 2\nu\int_0^T \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s \\
&\leq \frac{4c_3}{\nu}\int_0^T \|\boldsymbol{h}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s + \frac{4c_3}{\nu}\int_0^T \|\boldsymbol{v}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s + 2c_4\left(\int_0^T \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s\right)^2.
\end{aligned} \tag{7.18}$$

Set

$$T(R) := \sup\left\{T \geq 0 : \int_0^T \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s \leq \frac{\nu}{2c_4}\right\}$$

so that for all $T \in [0, T(R)]$ we have

$$\sup_{s\in[0,T]} \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{1/2}} + \nu \int_0^T \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s$$
$$\leq \frac{4c_3}{\nu}\int_0^T \|\boldsymbol{h}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s + \frac{4c_3}{\nu}\int_0^T \|\boldsymbol{v}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s. \tag{7.19}$$

We now seek a bound on the right-hand side: indeed, if we can find a time T_0 such that

$$\frac{4c_3}{\nu}\int_0^{T_0} \|\boldsymbol{h}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s + \frac{4c_3}{\nu}\int_0^{T_0} \|\boldsymbol{v}^R(s)\|^4_{\dot{H}^1}\,\mathrm{d}s < \frac{\nu^2}{2c_4}, \tag{7.20}$$

then $T(R) \geq T_0$. To see this, we proceed along the same lines as in the proof of Theorem 7.6: first note that

$$\int_0^{T(R)} \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s = \frac{\nu}{2c_4}$$

by continuity; but if $T(R) < T_0$ then (7.18) and (7.20) would imply that

$$\int_0^{T(R)} \|\boldsymbol{w}^R(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s < \frac{\nu}{2c_4},$$

which is a contradiction, and thus we must have $T(R) \geq T_0$.

First, let us find a bound for the $\boldsymbol{h}$ term. Applying $\Lambda^{1/2}$ to (7.14a) and taking the inner product with $\Lambda^{1/2}\boldsymbol{h}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{h}\|^2_{\dot{H}^{1/2}} + \nu\|\boldsymbol{h}\|^2_{\dot{H}^{3/2}} \leq 0.$$

For any $T > t > 0$, integrating in time over $[0, t]$ yields

$$\frac{1}{2}\|\boldsymbol{h}(t)\|^2_{\dot{H}^{1/2}} + \nu\int_0^t \|\boldsymbol{h}(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s \leq \frac{1}{2}\|\boldsymbol{u}_0\|^2_{\dot{H}^{1/2}},$$

and thus

$$\sup_{s\in[0,T]} \|\boldsymbol{h}(s)\|^2_{\dot{H}^{1/2}} + 2\nu\int_0^T \|\boldsymbol{h}(s)\|^2_{\dot{H}^{3/2}}\,\mathrm{d}s \leq 2\|\boldsymbol{u}_0\|^2_{\dot{H}^{1/2}}. \tag{7.21}$$

By interpolation,

$$\int_0^T \|\boldsymbol{h}(s)\|^4_{\dot{H}^1}\,\mathrm{d}s \leq \frac{2}{\nu}\|\boldsymbol{u}_0\|^4_{\dot{H}^{1/2}}. \tag{7.22}$$

Hence $\|\boldsymbol{h}(s)\|_{\dot{H}^1}^4$ is integrable on $[0,T]$, and thus we may choose T_1 such that

$$\int_0^{T_1} \|\boldsymbol{h}(s)\|_{\dot{H}^1}^4 \,\mathrm{d}s < \frac{\nu^3}{16c_3c_4}.$$

By the properties of the Stokes equation, this implies that

$$\int_0^{T_1} \|\boldsymbol{h}^R(s)\|_{\dot{H}^1}^4 \,\mathrm{d}s < \frac{\nu^3}{16c_3c_4} \tag{7.23}$$

for all $R > 0$. Note that, unlike the 2D case, T_1 really depends on the whole of $\boldsymbol{u}_0$.

Secondly, let us find a bound for the $\boldsymbol{v}$ term. Applying $\Lambda^{1/2}$ to (7.16a) and taking the inner product with $\Lambda^{1/2}\boldsymbol{v}^R$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^R\|_{\dot{H}^{1/2}}^2 + \nu\|\boldsymbol{v}^R\|_{\dot{H}^{3/2}}^2 \le \|(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R\|_{\dot{H}^{1/2}}\|\boldsymbol{v}^R\|_{\dot{H}^{1/2}} \le \|\boldsymbol{B}^R\|_{\dot{B}^{3/2}_{2,1}}^2\|\boldsymbol{v}^R\|_{\dot{H}^{1/2}}$$

by (7.4). Dropping the second term on the left-hand side yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^R\|_{\dot{H}^{1/2}} \le \|\boldsymbol{B}^R\|_{\dot{B}^{3/2}_{2,1}}^2.$$

For any $T > t > 0$, integrating in time over $[0,t]$ and taking the supremum over $t \in [0,T]$ yields

$$\sup_{s\in[0,T]} \|\boldsymbol{v}^R(s)\|_{\dot{H}^{1/2}} \le \int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \,\mathrm{d}s.$$

This implies that

$$\|\boldsymbol{v}^R(t)\|_{\dot{H}^{3/2}}^2 \le \frac{1}{\nu}\|\boldsymbol{B}^R(t)\|_{\dot{B}^{3/2}_{2,1}}^2 \int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \,\mathrm{d}s,$$

so that

$$\sup_{s\in[0,T]} \|\boldsymbol{v}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu\int_0^T \|\boldsymbol{v}^R(s)\|_{\dot{H}^{3/2}}^2 \,\mathrm{d}s \le 3\left(\int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \,\mathrm{d}s\right)^2. \tag{7.24}$$

Hence by interpolation,

$$\int_0^T \|\boldsymbol{v}^R(s)\|_{\dot{H}^1}^4 \,\mathrm{d}s \le \frac{1}{\nu}\left(\int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \,\mathrm{d}s\right)^4. \tag{7.25}$$

Now, let $T \le T_1$ be any time such that assumption (7.12) holds. Then we obtain

$$\int_0^T \|\boldsymbol{v}^R(s)\|_{\dot{H}^1}^4 \, \mathrm{d}s \le \frac{\nu^3}{16c_3c_4}. \tag{7.26}$$

Combining (7.23) and (7.26) yields (7.20) with $T_0 = T$, and hence $T(R) \ge T$ for all such T; in particular, $T(R) \ge T_1$.

Moreover, (7.19) holds on the interval $[0, T]$, and substituting (7.22) and (7.25) into (7.19) yields

$$\begin{aligned} \sup_{s\in[0,T]} \|\boldsymbol{w}^R(s)\|_{\dot{H}^{1/2}}^2 + \nu \int_0^T \|\boldsymbol{w}^R(s)\|_{\dot{H}^{3/2}}^2 \, \mathrm{d}s \\ \le \frac{8c_3}{\nu^2}\|\boldsymbol{u}_0\|_{\dot{H}^{1/2}}^4 + \frac{4c_3}{\nu^2}\left(\int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \, \mathrm{d}s\right)^4. \end{aligned} \tag{7.27}$$

Hence, using (7.21), (7.24) and (7.27), we obtain

$$\begin{aligned} &\sup_{s\in[0,T]} \|\boldsymbol{u}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \boldsymbol{u}^R(s)\|_{\dot{H}^{1/2}}^2 \, \mathrm{d}s \\ &\quad\le \sup_{s\in[0,T]} \|\boldsymbol{h}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \boldsymbol{h}^R(s)\|_{\dot{H}^{1/2}}^2 \, \mathrm{d}s \\ &\qquad + \sup_{s\in[0,T]} \|\boldsymbol{v}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \boldsymbol{v}^R(s)\|_{\dot{H}^{1/2}}^2 \, \mathrm{d}s \\ &\qquad + \sup_{s\in[0,T]} \|\boldsymbol{w}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \boldsymbol{w}^R(s)\|_{\dot{H}^{1/2}}^2 \, \mathrm{d}s \\ &\quad\le 2\|\boldsymbol{u}_0\|_{\dot{H}^{1/2}}^2 + \frac{16c_3}{\nu^2}\|\boldsymbol{u}_0\|_{\dot{H}^{1/2}}^4 \\ &\qquad + 3\left(\int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \, \mathrm{d}s\right)^2 + \frac{8c_3}{\nu^2}\left(\int_0^T \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \, \mathrm{d}s\right)^4. \end{aligned}$$

This completes the proof. □

Proposition 7.7 appears to show that the existence time for the $\boldsymbol{u}$ equation depends on the existence time for the $\boldsymbol{B}$ equation; but it is clear from Proposition 7.2 that the existence time for the $\boldsymbol{B}$ equation ought to depend on the existence time for the $\boldsymbol{u}$ equation. In order to circumvent the impending doom of this seemingly circular argument, we now show that there is some (short) time interval such that (7.12) holds for all $R > 0$.

Lemma 7.8. *There is a time* $T_2 = T_2(\nu, \|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}}, \|\boldsymbol{B}_0\|_{\dot{B}^{3/2}_{2,1}}) > 0$ *such that*

$$\int_0^T \|\boldsymbol{B}^R(s)\|^2_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s \leq \frac{\nu}{2(c_3c_4)^{1/4}} =: C_*$$

for all $T \leq \min\{T_1, T_2\}$ *and all* $R > 0$.

Proof. Define

$$Z_R(t) := \int_0^t \|\boldsymbol{B}^R(s)\|^2_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s.$$

Using the estimate on $\|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}$ from Proposition 7.2, we obtain

$$Z_R(t) \leq t\|\boldsymbol{B}_0\|^2_{\dot{B}^{3/2}_{2,1}} \exp\left(2c_1 \int_0^t \|\nabla \boldsymbol{u}(s)\|_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s\right).$$

Using the estimate on $\int_0^t \|\nabla \boldsymbol{u}(s)\|_{\dot{B}^{3/2}_{2,1}}\,\mathrm{d}s$ from Proposition 7.4, we obtain

$$Z_R(t) \leq t\|\boldsymbol{B}_0\|^2_{\dot{B}^{3/2}_{2,1}} \exp\left(\frac{2c_1}{\nu}\|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}} + \frac{2c_1c_2}{\nu}\int_0^t \|\nabla \boldsymbol{u}^R(s)\|^2_{\dot{H}^{1/2}}\,\mathrm{d}s + \frac{2c_1c_2}{\nu} Z_R(t)\right). \tag{7.28}$$

Recall from (7.12) that $C_* := \frac{\nu}{2(c_3c_4)^{1/4}}$. Let

$$\begin{aligned} T_2 := \frac{C_*}{\|\boldsymbol{B}_0\|^2_{\dot{B}^{3/2}_{2,1}}} \exp\Bigg(&-\frac{2c_1}{\nu}\|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}} - \frac{2c_1c_2}{\nu^2}\|\boldsymbol{u}_0\|^2_{\dot{B}^{1/2}_{2,1}} - \frac{8c_1c_2c_3}{\nu^4}\|\boldsymbol{u}_0\|^4_{\dot{B}^{1/2}_{2,1}} \\ &- \frac{3c_1c_2}{\nu^2}C_* - \frac{2c_1c_2}{\nu}C_*^2 - \frac{4c_1c_2c_3}{\nu^4}C_*^4\Bigg). \end{aligned}$$

Suppose $t < \min\{T_1, T_2\}$ and $Z_R(t) \leq C_*$. Then using Proposition 7.7 to estimate the term $\int_0^t \|\nabla \boldsymbol{u}^R(s)\|^2_{\dot{H}^{1/2}}\,\mathrm{d}s$, from (7.28) we obtain

$$\begin{aligned} Z_R(t) &\leq t\|\boldsymbol{B}_0\|^2_{\dot{B}^{3/2}_{2,1}} \exp\Bigg(\frac{2c_1}{\nu}\|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}} + \frac{2c_1c_2}{\nu^2}\|\boldsymbol{u}_0\|^2_{\dot{B}^{1/2}_{2,1}} + \frac{16c_1c_2c_3}{\nu^4}\|\boldsymbol{u}_0\|^4_{\dot{B}^{1/2}_{2,1}} \\ &\qquad + \frac{3c_1c_2}{\nu^2}Z_R(t) + \frac{2c_1c_2}{\nu}[Z_R(t)]^2 + \frac{8c_1c_2c_3}{\nu^4}[Z_R(t)]^4\Bigg) \\ &\leq t\|\boldsymbol{B}_0\|^2_{\dot{B}^{3/2}_{2,1}} \exp\Bigg(\frac{2c_1}{\nu}\|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}} + \frac{2c_1c_2}{\nu^2}\|\boldsymbol{u}_0\|^2_{\dot{B}^{1/2}_{2,1}} + \frac{16c_1c_2c_3}{\nu^4}\|\boldsymbol{u}_0\|^4_{\dot{B}^{1/2}_{2,1}} \\ &\qquad + \frac{3c_1c_2}{\nu^2}C_* + \frac{2c_1c_2}{\nu}C_*^2 + \frac{8c_1c_2c_3}{\nu^4}C_*^4\Bigg) \\ &< C_*. \end{aligned}$$

As $Z_R(t)$ is continuous and $Z_R(0) = 0$, this means that $Z_R(t)$ can never equal C_* as long as $0 \le t < \min\{T_1, T_2\}$, and hence $Z_R(t) < C_*$ for all $0 \le t < \min\{T_1, T_2\}$. $\square$

Combining the energy estimate (7.9) with Proposition 7.7 and Lemma 7.8, we obtain the following bound on $\int_0^t \|\boldsymbol{u}^R(s)\|_{H^{3/2}}^2 \, \mathrm{d}s$:

$$\begin{aligned}
\int_0^t \|\boldsymbol{u}^R(s)\|_{H^{3/2}}^2 \, \mathrm{d}s &\le \int_0^t \|\boldsymbol{u}^R(s)\|_{L^2}^2 \, \mathrm{d}s + \int_0^t \|\nabla \boldsymbol{u}^R(s)\|_{\dot{H}^{1/2}}^2 \, \mathrm{d}s \\
&\le 2t(\|\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{B}_0\|_{L^2}^2) + \frac{1}{\nu}\|\boldsymbol{u}_0\|_{\dot{H}^{1/2}}^2 + \frac{8c_3}{\nu^3}\|\boldsymbol{u}_0\|_{\dot{H}^{1/2}}^4 \\
&\quad + \frac{3}{2\nu}\left(\int_0^t \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \, \mathrm{d}s\right)^2 + \frac{4c_3}{\nu^3}\left(\int_0^t \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \, \mathrm{d}s\right)^4
\end{aligned} \tag{7.29}$$

for all $0 \le t \le \min\{T_1, T_2\}$.

We can now proceed analogously to the 2D case and show that $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ are uniformly bounded in the corresponding Besov spaces, although the algebra is slightly more involved.

Theorem 7.9. *There is a time $T_* = T_*(\nu, \boldsymbol{u}_0, \|\boldsymbol{B}_0\|_{B^{3/2}_{2,1}}) > 0$ such that*

$$\begin{aligned}
&\boldsymbol{u}^R \textit{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)) \cap L^1(0, T_*; \dot{B}^{5/2}_{2,1}(\mathbb{R}^3)), \\
&\boldsymbol{B}^R \textit{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)).
\end{aligned}$$

Proof. Let

$$\begin{aligned}
M_1 &= \|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}} + \frac{c_2}{\nu}\|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}}^2 + \frac{8c_2c_3}{\nu^3}\|\boldsymbol{u}_0\|_{\dot{B}^{1/2}_{2,1}}^4 \\
M_2 &= 2c_2(\|\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{B}_0\|_{L^2}^2).
\end{aligned}$$

Substituting from equation (7.29) into Proposition 7.4, when $t \le \min\{T_1, T_2\}$ we obtain

$$\begin{aligned}
&\|\boldsymbol{u}^R(t)\|_{\dot{B}^{1/2}_{2,1}} + \nu \int_0^t \|\nabla \boldsymbol{u}^R(s)\|_{\dot{B}^{3/2}_{2,1}} \, \mathrm{d}s \\
&\qquad \le M_1 + M_2 t + c_2 Z_R(t) + \frac{3c_2}{2\nu}(Z_R(t))^2 + \frac{4c_2c_3}{\nu^3}(Z_R(t))^4,
\end{aligned}$$

where $Z_R(t) := \int_0^t \|\boldsymbol{B}^R(s)\|_{\dot{B}^{3/2}_{2,1}}^2 \, \mathrm{d}s$ as above. Letting $M_3 = \|\boldsymbol{B}_0\|_{\dot{B}^{3/2}_{2,1}}^2$, Proposition 7.2 yields

$$
\begin{aligned}
Z_R(t) &\le M_3 \int_0^t \exp\left(2c_1 \int_0^\tau \|\nabla \boldsymbol{u}^R(s)\|_{\dot{B}^{3/2}_{2,1}} \,\mathrm{d}s\right) \mathrm{d}\tau \\
&\le M_3 t \exp\left(2c_1 \int_0^t \|\nabla \boldsymbol{u}^R(s)\|_{\dot{B}^{3/2}_{2,1}} \,\mathrm{d}s\right).
\end{aligned}
$$

Setting

$$
\begin{aligned}
X_R(t) &= \|\boldsymbol{u}^R(t)\|_{\dot{B}^{1/2}_{2,1}}, \\
Y_R(t) &= \nu \int_0^t \|\nabla \boldsymbol{u}^R(s)\|_{\dot{B}^{3/2}_{2,1}} \,\mathrm{d}s,
\end{aligned}
$$

yields

$$
\begin{aligned}
X_R(t) + Y_R(t) \le M_1 + M_2 t &+ c_2 M_3 t \exp(2c_1 Y_R(t)/\nu) \\
&+ \frac{3c_2}{2\nu} M_3^2 t^2 \exp(4c_1 Y_R(t)/\nu) + \frac{4c_2 c_3}{\nu^3} M_3^4 t^4 \exp(8c_1 Y_R(t)/\nu).
\end{aligned} \tag{7.30}
$$

Let

$$
M_4 = (2 + c_2) M_1 + \frac{3c_2}{2\nu} M_1^2 + \frac{4c_2 c_3}{\nu^3} M_1^4,
$$

and set

$$
T_* = \min\left\{ T_1, T_2, \frac{M_1}{M_2}, \frac{M_1}{M_3} \exp(-2c_1 M_4/\nu) \right\}.
$$

It suffices to show that $X_R(t) + Y_R(t) \le M_4$ for all $t \in [0, T_*)$ and all $R > 0$. To see this, note that $Y_R(t)$ is continuous and $Y_R(0) = 0$. Now, suppose $t < T_*$ and $Y_R(t) \le M_4$; then

$$
\begin{aligned}
Y_R(t) &\le M_1 + M_2 t + c_2 M_3 t \exp(2c_1 Y_R(t)/\nu) \\
&\qquad + \frac{3c_2}{2\nu} M_3^2 t^2 \exp(4c_1 Y_R(t)/\nu) + \frac{4c_2 c_3}{\nu^3} M_3^4 t \exp(8c_1 Y_R(t)/\nu) \\
&< M_1 + M_1 + c_2 M_1 \exp\left(\frac{2c_1}{\nu}[Y_R(t) - M_4]\right) \\
&\qquad + \frac{3c_2}{2\nu} M_1^2 \exp\left(\frac{4c_1}{\nu}[Y_R(t) - M_4]\right) + \frac{4c_2 c_3}{\nu^3} M_1^4 \exp\left(\frac{8c_1}{\nu}[Y_R(t) - M_4]\right) \\
&\le M_4.
\end{aligned}
$$

This means that $Y_R(t)$ can never equal M_4 on the interval $[0, T_*)$, hence $Y_R(t) < M_4$ for all $t \in [0, T_*)$. The result follows from inequality (7.30) and Proposition 7.2. $\square$

Notice that, in the 3D case, T_1 (and hence T_*) depends on the whole of $\boldsymbol{u}_0$ and not just on the norm $\|\boldsymbol{u}_0\|_{B^{1/2}_{2,1}}$.

7.3 Existence Proof

In summary, in either the 2D or the 3D case, there is some time T_* such that

$$\boldsymbol{u}^R \text{ is uniformly bounded in } L^\infty(0,T_*;\dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0,T_*;\dot{B}^{n/2+1}_{2,1}(\mathbb{R}^n)), \tag{7.31a}$$

$$\boldsymbol{B}^R \text{ is uniformly bounded in } L^\infty(0,T_*;\dot{B}^{n/2}_{2,1}(\mathbb{R}^n)). \tag{7.31b}$$

Having obtained these uniform bounds, in this section we outline the proof of Theorem 7.1, using broadly the same method as in Section 4.2.2 to show the existence of a weak solution.

Let us first note that since the initial data is taken in inhomogeneous Besov spaces, the standard energy estimate (7.9) implies that $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ are uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}^n))$ for any $T>0$, and hence the uniform bounds (7.31) imply that

$$\boldsymbol{u}^R \text{ is uniformly bounded in } L^\infty(0,T_*;B^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0,T_*;B^{n/2+1}_{2,1}(\mathbb{R}^n)), \tag{7.32a}$$

$$\boldsymbol{B}^R \text{ is uniformly bounded in } L^\infty(0,T_*;B^{n/2}_{2,1}(\mathbb{R}^n)). \tag{7.32b}$$

7.3.1 Bounds on the Time Derivatives

We first obtain uniform bounds on the time derivatives $\frac{\partial \boldsymbol{u}^R}{\partial t}$ and $\frac{\partial \boldsymbol{B}^R}{\partial t}$. By first applying the Leray projector Π to the equations, we may eliminate the pressure term in (7.7) and consider the equations

$$\frac{\partial \boldsymbol{u}^R}{\partial t} - \nu\Pi\Delta\boldsymbol{u}^R = \mathcal{S}_R\Pi[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R] - \mathcal{S}_R\Pi[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R], \tag{7.33a}$$

$$\frac{\partial \boldsymbol{B}^R}{\partial t} = \mathcal{S}_R\Pi[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{u}^R] - \mathcal{S}_R\Pi[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R]. \tag{7.33b}$$

Taking the $\dot{B}^{n/2-1}_{2,1}$ norm of both sides of (7.33a) yields

$$\begin{aligned}\left\|\frac{\partial \boldsymbol{u}^R}{\partial t}\right\|_{\dot{B}^{n/2-1}_{2,1}} &\leq \nu \left\|\Delta \boldsymbol{u}^R\right\|_{\dot{B}^{n/2-1}_{2,1}} + \left\|(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R\right\|_{\dot{B}^{n/2-1}_{2,1}} + \left\|(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{u}^R\right\|_{\dot{B}^{n/2-1}_{2,1}} \\ &\leq \nu \underbrace{\left\|\boldsymbol{u}^R\right\|_{\dot{B}^{n/2+1}_{2,1}}}_{\in L^1(0,T_*)} + \underbrace{\left\|\boldsymbol{B}^R\right\|^2_{\dot{B}^{n/2}_{2,1}}}_{\in L^\infty(0,T_*)} + \underbrace{\left\|\boldsymbol{u}^R\right\|^2_{\dot{B}^{n/2}_{2,1}}}_{\in L^1(0,T_*)}\end{aligned}$$

where we have used the fact that, by interpolation, the uniform bounds (7.31) imply that $\boldsymbol{u}^R$ is uniformly bounded in $L^2(0,T_*;\dot{B}^{n/2}_{2,1}(\mathbb{R}^n))$. Similarly, taking the $\dot{B}^{n/2-1}_{2,1}$ norm of both sides of (7.33b) yields

$$\left\|\frac{\partial \boldsymbol{B}^R}{\partial t}\right\|_{\dot{B}^{n/2-1}_{2,1}} \leq 2\underbrace{\left\|\boldsymbol{B}^R\right\|_{\dot{B}^{n/2}_{2,1}}}_{\in L^\infty(0,T_*)} \cdot \underbrace{\left\|\boldsymbol{u}^R\right\|_{\dot{B}^{n/2}_{2,1}}}_{\in L^2(0,T_*)}.$$

Hence

$$\frac{\partial \boldsymbol{u}^R}{\partial t} \text{ is uniformly bounded in } L^1(0,T_*;\dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n)), \tag{7.34a}$$
$$\frac{\partial \boldsymbol{B}^R}{\partial t} \text{ is uniformly bounded in } L^2(0,T_*;\dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n)). \tag{7.34b}$$

Repeating these bounds using the inhomogeneous norms and (7.32) implies that

$$\frac{\partial \boldsymbol{u}^R}{\partial t} \text{ is uniformly bounded in } L^1(0,T_*;B^{n/2-1}_{2,1}(\mathbb{R}^n)), \tag{7.35a}$$
$$\frac{\partial \boldsymbol{B}^R}{\partial t} \text{ is uniformly bounded in } L^2(0,T_*;B^{n/2-1}_{2,1}(\mathbb{R}^n)). \tag{7.35b}$$

7.3.2 Strong Convergence

Using the uniform bounds (7.32) and (7.35), one may use the Banach–Alaoglu theorem to extract a weakly-$*$ convergent subsequence such that

$$\begin{aligned}\boldsymbol{u}^{R_m} &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{in } L^\infty(0,T_*;B^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0,T_*;B^{n/2+1}_{2,1}(\mathbb{R}^n)),\\ \boldsymbol{B}^{R_m} &\overset{*}{\rightharpoonup} \boldsymbol{B} && \text{in } L^\infty(0,T_*;B^{n/2}_{2,1}(\mathbb{R}^n)),\\ \frac{\partial \boldsymbol{u}^{R_m}}{\partial t} &\overset{*}{\rightharpoonup} \frac{\partial \boldsymbol{u}}{\partial t} && \text{in } L^1(0,T_*;B^{n/2-1}_{2,1}(\mathbb{R}^n)),\\ \frac{\partial \boldsymbol{B}^{R_m}}{\partial t} &\overset{*}{\rightharpoonup} \frac{\partial \boldsymbol{B}}{\partial t} && \text{in } L^2(0,T_*;B^{n/2-1}_{2,1}(\mathbb{R}^n)).\end{aligned}$$

We now show that $(\boldsymbol{u}, \boldsymbol{B})$ is a *weak* solution of the equations. By embedding the Besov spaces $B^s_{2,1}$ in the corresponding Sobolev spaces H^s, and using the variant of the Aubin–Lions compactness lemma from Section 4.2.2 (see Proposition 2.7 in Chemin et al. (2006)), there exists a subsequence of $(\boldsymbol{u}^{R_m}, \boldsymbol{B}^{R_m})$ that converges strongly in $L^2(0,T;H^s(K))$ for any $s \in (\frac{n}{2}-1, \frac{n}{2})$ and any compact subset $K \subset \mathbb{R}^n$; and thus they also converge strongly in $L^2(0,T;L^2(K))$, and hence the limit satisfies

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;V(\mathbb{R}^n)).$$

This local strong convergence allows us to pass to the limit in the nonlinear terms: an argument similar to Proposition 4.7 will show that (after passing to a subsequence)

$$\mathcal{S}_{R_m}[(\boldsymbol{u}^{R_m} \cdot \nabla)\boldsymbol{B}^{R_m}] \overset{*}{\rightharpoonup} (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}$$

(and so on) in $L^2(0,T;V^*(\mathbb{R}^n))$ (see §2.2.4 of Chemin et al. (2006) for full details). Thus $(\boldsymbol{u}, \boldsymbol{B})$ is indeed a weak solution of (1.5).

7.3.3 Uniqueness in 3D

We now prove a uniqueness result in 3D, in a very similar manner to Proposition 6.6.

Proposition 7.10. *Let $(\boldsymbol{u}_j, \boldsymbol{B}_j)$, $j = 1, 2$, be two solutions of* (1.5) *with the same initial conditions $\boldsymbol{u}_j(0) = \boldsymbol{u}_0$, $\boldsymbol{B}_j(0) = \boldsymbol{B}_0$, such that*

$$\begin{aligned} \boldsymbol{u}_j &\in L^\infty(0,T_*;B^{1/2}_{2,1}(\mathbb{R}^3)) \cap L^1(0,T_*;B^{5/2}_{2,1}(\mathbb{R}^3)), \\ \boldsymbol{B}_j &\in L^\infty(0,T_*;B^{3/2}_{2,1}(\mathbb{R}^3)). \end{aligned}$$

Then $(\boldsymbol{u}_1, \boldsymbol{B}_1) = (\boldsymbol{u}_2, \boldsymbol{B}_2)$ as functions in $L^\infty(0,T;L^2(\mathbb{R}^3))$.

Proof. Take the equations for $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2)$ and subtract: writing $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{z} = \boldsymbol{B}_1 - \boldsymbol{B}_2$ and $q = p_1 - p_2$, we obtain

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_2 - \nu\Delta\boldsymbol{w} + \nabla q = (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{z} + (\boldsymbol{z} \cdot \nabla)\boldsymbol{B}_2, \tag{7.36a}$$

$$\frac{\partial \boldsymbol{z}}{\partial t} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{z} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{B}_2 = (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{w} + (\boldsymbol{z} \cdot \nabla)\boldsymbol{u}_2. \tag{7.36b}$$

Taking the inner product of (7.36a) with $\boldsymbol{w}$ and (7.36b) with $\boldsymbol{z}$, and adding, yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{w}\|_{L^2}^2+\|\boldsymbol{z}\|_{L^2}^2\right)+\nu\|\nabla\boldsymbol{w}\|_{L^2}^2\\
&\qquad=\langle(\boldsymbol{z}\cdot\nabla)\boldsymbol{B}_2,\boldsymbol{w}\rangle-\langle(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_2,\boldsymbol{w}\rangle+\langle(\boldsymbol{z}\cdot\nabla)\boldsymbol{u}_2,\boldsymbol{z}\rangle-\langle(\boldsymbol{w}\cdot\nabla)\boldsymbol{B}_2,\boldsymbol{z}\rangle\\
&\qquad\le\|\boldsymbol{z}\|_{L^2}\|\nabla\boldsymbol{w}\|_{L^2}\|\boldsymbol{B}_2\|_{L^\infty}+\|\boldsymbol{w}\|_{L^2}\|\nabla\boldsymbol{w}\|_{L^2}\|\boldsymbol{u}_2\|_{L^\infty}\\
&\qquad\qquad+\|\boldsymbol{z}\|_{L^2}^2\|\nabla\boldsymbol{u}_2\|_{L^\infty}+\|\boldsymbol{w}\|_{L^6}\|\nabla\boldsymbol{B}_2\|_{L^3}\|\boldsymbol{z}\|_{L^2}\\
&\qquad\le\left(\|\boldsymbol{w}\|_{L^2}+\|\boldsymbol{z}\|_{L^2}\right)\|\nabla\boldsymbol{w}\|_{L^2}\left(\|\boldsymbol{u}_2\|_{\dot{B}^{3/2}_{2,1}}+\|\boldsymbol{B}_2\|_{\dot{B}^{3/2}_{2,1}}\right)+\|\boldsymbol{z}\|_{L^2}^2\|\nabla\boldsymbol{u}_2\|_{\dot{B}^{3/2}_{2,1}},
\end{aligned}$$

so by Young's inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{w}\|_{L^2}^2+\|\boldsymbol{z}\|_{L^2}^2\right)+\nu\|\nabla\boldsymbol{w}\|_{L^2}^2\\
&\qquad\le\frac{c}{\nu}\left(\|\boldsymbol{u}_2\|_{\dot{B}^{3/2}_{2,1}}^2+\|\boldsymbol{B}_2\|_{\dot{B}^{3/2}_{2,1}}^2+\|\nabla\boldsymbol{u}_2\|_{\dot{B}^{3/2}_{2,1}}\right)\left(\|\boldsymbol{w}\|_{L^2}^2+\|\boldsymbol{z}\|_{L^2}^2\right)
\end{aligned}$$

and uniqueness follows by Gronwall's inequality. □

Note, however, that this argument does not apply in 2D. This is because the term $\langle(\boldsymbol{w}\cdot\nabla)\boldsymbol{B}_2,\boldsymbol{z}\rangle$ cannot be estimated in the same way: in 3D we used the inequality

$$|\langle(\boldsymbol{w}\cdot\nabla)\boldsymbol{B}_2,\boldsymbol{z}\rangle|\le\|\boldsymbol{w}\|_{L^6}\|\nabla\boldsymbol{B}_2\|_{L^3}\|\boldsymbol{z}\|_{L^2}\le\|\nabla\boldsymbol{w}\|_{L^2}\|\boldsymbol{B}_2\|_{\dot{B}^{3/2}_{2,1}}\|\boldsymbol{z}\|_{L^2},$$

but in 2D the best we can do is

$$|\langle(\boldsymbol{w}\cdot\nabla)\boldsymbol{B}_2,\boldsymbol{z}\rangle|\le\|\boldsymbol{w}\|_{L^\infty}\|\nabla\boldsymbol{B}_2\|_{L^2}\|\boldsymbol{z}\|_{L^2}\le\|\boldsymbol{w}\|_{L^\infty}\|\boldsymbol{B}_2\|_{\dot{B}^{1}_{2,1}}\|\boldsymbol{z}\|_{L^2},$$

since the embedding $H^1\hookrightarrow L^\infty$ fails to hold in 2D. While we could use the embedding $\dot{B}^1_{2,1}\hookrightarrow L^\infty$, that would not allow us to absorb the term into the $\|\nabla\boldsymbol{w}\|_{L^2}$ term on the left-hand side.

This leaves us in the odd situation where we can prove uniqueness in 3D, but not in 2D! More importantly, however, it shows that a proof along the lines of Chapter 6 would not necessarily work, since the uniqueness proof is just a simpler version of the proof that the truncated solutions $(\boldsymbol{u}^R,\boldsymbol{B}^R)$ are Cauchy in $L^\infty(0,T;L^2(\mathbb{R}^n))$ (see Proposition 6.3).

Chapter 8

Conclusion and Open Problems

This thesis has focussed on existence and uniqueness theory for two main systems of PDEs related to MHD.

Stokes-MHD: Let us consider first the Stokes-MHD system

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.2a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.2b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.2c}$$

with $\eta > 0$. In Chapters 3 and 4, we proved existence, uniqueness and regularity of weak solutions to (1.2) in 2D, and existence of weak solutions in 3D. This involved a generalisation of Ladyzhenskaya's inequality to involve the weak L^2 space $L^{2,\infty}$:

$$\|f\|_{L^4} \le c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|_{L^2}^{1/2}.$$

Combining this with elliptic regularity for the Stokes equations, we proved the existence of a unique weak solution $(\boldsymbol{u},\boldsymbol{B})$ of (1.2) such that

$$\begin{aligned}
\boldsymbol{u} &\in L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),\\
\boldsymbol{B} &\in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),\\
\frac{\partial\boldsymbol{B}}{\partial t} &\in L^2(0,T;H^{-1}(\Omega)),
\end{aligned}$$

for a domain Ω in 2D, provided the initial data $\boldsymbol{B}_0 \in L^2(\Omega)$. Much like the Navier–Stokes equations, one can prove existence of weak solutions in 3D (though the exponents get a little messier), but not uniqueness.

MHD: For the non-resistive MHD equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.5a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.5b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.5c}$$

the previously best known existence result, due to Jiu & Niu (2006), asserted the local existence of solutions in 2D for initial data in H^s with integer $s \geq 3$. In Chapters 5 and 6, we established the local existence and uniqueness of a solution $\boldsymbol{u}, \boldsymbol{B} \in C([0,T_*]; H^s(\mathbb{R}^n))$ in 2D and 3D for initial data $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^s(\mathbb{R}^n)$ for any $s > n/2$. The results also apply to the non-resistive Stokes-MHD system (1.6).

The commutator estimate

$$\|\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] - (\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B})\|_{L^2} \leq c\|\nabla\boldsymbol{u}\|_{H^s}\|\boldsymbol{B}\|_{H^s}$$

(see Theorem 5.1) formed a key part of the a priori estimates. In Section 5.2 we exhibited a counterexample to the commutator estimate in the case $s = 1$ in 2D. Since this counterexample depended heavily on having just one derivative, it is possible that the inequality may in fact be true for $s = 3/2$ in 3D. In any case, it remains to be seen whether existence and uniqueness of solutions in $H^{n/2}$ could be proved in some other fashion.

It would also be interesting to investigate whether existence of solutions in *homogeneous* Sobolev spaces $\dot{H}^s$ for $s > n/2$ could be proven, though it should be noted that these spaces are no longer Banach spaces.

Besov spaces: The fundamental reason why establishing existence of solutions in $H^{n/2}$ is more difficult is the failure of the Sobolev embedding $H^{n/2} \hookrightarrow L^\infty$. We therefore turned instead to spaces with the same scaling which do embed in L^∞, in particular the Besov space $B^{n/2}_{2,1}$. In Chapter 7, we consider equations (1.5) in Besov spaces. With initial data $\boldsymbol{u}_0 \in B^{n/2-1}_{2,1}(\mathbb{R}^n)$ and $\boldsymbol{B}_0 \in B^{n/2}_{2,1}(\mathbb{R}^n)$ for $n = 2, 3$, we proved the existence of a solution $(\boldsymbol{u}, \boldsymbol{B})$ satisfying

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T_*; B^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0,T_*; B^{n/2+1}_{2,1}(\mathbb{R}^n)),\\
&\boldsymbol{B} \in L^\infty(0,T_*; B^{n/2}_{2,1}(\mathbb{R}^n)).
\end{aligned}$$

It is clear, however, that there is considerable scope for further work. The a priori estimates in Section 7.1, valid in both 2D and 3D, depend only on the norms of the initial data in the corresponding *homogeneous* Besov spaces, that is $\|\boldsymbol{u}_0\|_{\dot{B}^{n/2-1}_{2,1}}$ and $\|\boldsymbol{B}_0\|_{\dot{B}^{n/2}_{2,1}}$. However, the estimate for the $\boldsymbol{u}$ equation includes the term

$$\int_0^t \|\boldsymbol{u}(s)\|^2_{H^{n/2}}\,\mathrm{d}s$$

on the right-hand side, which arises from the use of the commutator estimate Lemma 7.5, taken from Chemin (1992):

$$\langle \Lambda^{n/2-1}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}], \Lambda^{n/2-1}\boldsymbol{u}\rangle \le c\|\boldsymbol{u}\|_{H^{n/2}}\|\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\boldsymbol{u}\|_{\dot{H}^{n/2-1}}.$$

It is natural to ask whether all three norms on the right-hand side could be taken in homogeneous spaces. In Appendix A we prove a partial generalisation of Lemma 7.5: namely that

$$|\langle \Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}], \Lambda^s\boldsymbol{u}\rangle| \le c\|\boldsymbol{u}\|_{\dot{H}^{s_1}}\|\boldsymbol{u}\|_{\dot{H}^{s_2}}\|\boldsymbol{u}\|_{\dot{H}^{s}}.$$

provided that $s \ge 1$ and $s_1, s_2 > 0$ such that

$$1 \le s_1 < \tfrac{n}{2}+1 \qquad \text{and} \qquad s_1+s_2 = s+\tfrac{n}{2}+1.$$

Unfortunately the case we would want to apply requires $s = n/2-1$, which does not satisfy $s \ge 1$ in 2D or 3D.

If such a generalisation could be proved, then the a priori estimates would instead include the term

$$\int_0^t \|\nabla\boldsymbol{u}(s)\|^2_{\dot{H}^{n/2-1}}\,\mathrm{d}s$$

on the right-hand side. In 2D this would still be most easily dealt with using the energy estimate (7.9) by assuming that $\boldsymbol{B}_0 \in B^1_{2,1} = \dot{B}^1_{2,1}\cap L^2$, though it should be noted that $\boldsymbol{u}_0 \in \dot{B}^0_{2,1}$ suffices thanks to the embedding $\dot{B}^0_{2,1} \hookrightarrow L^2$ (see Proposition 2.10). In 3D, however, Proposition 7.7 and Lemma 7.8 would allow the a priori estimates to be closed up while assuming only that $\boldsymbol{u}_0 \in \dot{B}^{1/2}_{2,1}$ and $\boldsymbol{B}_0 \in \dot{B}^{3/2}_{2,1}$.

Nonetheless, even though it seems that it would be relatively straightforward to generalise the a priori estimates to homogeneous Besov spaces, obtaining a bona fide solution of the equations (1.5) without assuming that the initial conditions have finite energy appears rather more difficult. The most natural sense would be to show that $(\boldsymbol{u}, \boldsymbol{B})$ solve equations (1.5) as an equality in $L^1(0, T_*; \dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n))$ (since that is where the time derivatives lie).

However, it is not immediately obvious how to proceed, not least because $\dot{B}^s_{2,1}$ is only a Banach space for $s \le n/2$ (in contrast, all inhomogeneous Besov spaces are Banach spaces). Furthermore, it is not clear which sense of strong convergence would be required to guarantee convergence of the nonlinear terms.

Most urgently, however, it remains to prove that the solution whose existence is asserted in Theorem 7.1 is unique in 2D. While it might be possible to adapt the proofs of the a priori estimates (Propositions 7.2 and 7.4) to yield a proof of uniqueness by working instead in the space $B^0_{2,1}$, the proof relies on certain cancellations which are no longer available in the proof of uniqueness, and initial investigations suggest that such an approach will likely not succeed.

An alternative approach would be to recast the equations in a Lagrangian formulation and consider the particle trajectories of the magnetic field $\boldsymbol{B}$. The Lagrangian approach, most notably applied to the Euler equations by Yudovich (1963), has yielded significant results in Besov spaces for both the Euler equations (due to Chae (2004)) and for MHD.

In particular, in proving existence and uniqueness of solutions to fully ideal MHD in the Besov space $B^{1+n/p}_{p,1}(\mathbb{R}^n)$, Miao & Yuan (2006) use the volume-preservation of the push-forward along particle trajectories of $\boldsymbol{u}+\boldsymbol{B}$ and $\boldsymbol{u}-\boldsymbol{B}$ to yield uniqueness; such a method could perhaps be adapted to the non-resistive case (though the presence of the diffusion term in $\boldsymbol{u}$ might complicate the proof somewhat).

More broadly, it would be interesting to see if the results of Chapters 6 and 7 could be generalised to $W^{s,p}$ (for $s > n/p$) and $B^{n/p}_{p,1}$ respectively. In particular, the original commutator estimate of Kato & Ponce (1988) is valid for $1 < p < \infty$, and while the proof of Theorem 5.1 uses some of the properties of L^2, it may prove possible to adapt the proof to $p \neq 2$.

However, generalising Theorem 7.1 to, for example, $\boldsymbol{u}_0 \in B^{n/p-1}_{p,1}(\mathbb{R}^n)$ and $\boldsymbol{B}_0 \in B^{n/p}_{p,1}(\mathbb{R}^n)$ is trickier. The a priori estimate for $\boldsymbol{u}$ (Proposition 7.4) depends fundamentally on the fact that if $\boldsymbol{u}$ solves the heat equation with initial data $\boldsymbol{u}_0 \in \dot{B}^s_{2,1}(\mathbb{R}^n)$, then $\boldsymbol{u} \in L^1(0,T;\dot{B}^{s+2}_{2,1}(\mathbb{R}^n))$, which specifically requires $p = 2$.

Nonetheless, more recent results for related equations in Besov spaces — of which Chae (2004) and Miao & Yuan (2006) are but two — do not require $p = 2$. One key step would be to generalise the commutator estimate Lemma 7.5 to the case $p \neq 2$, perhaps along similar lines to the proof in Appendix A.

One final point is that the result (Theorem 7.1) in Besov spaces requires one fewer derivative for the initial condition $\boldsymbol{u}_0$, due to the smoothing effect of the diffusion term in the $\boldsymbol{u}$ equation. It remains to be seen whether this is specific to Besov spaces, or whether Theorem 6.1 could be generalised to prove local existence of solutions for initial data $\boldsymbol{u}_0 \in H^{s-1}$ and $\boldsymbol{B}_0 \in H^s$ for $s > n/2$.

Magnetic relaxation: The ultimate motivation behind much of this thesis has been to lay the necessary foundations for a rigorous study of magnetic relaxation. As mentioned in the introduction, the long-time limit of any physically reasonable non-resistive MHD system ought to converge to an equilibrium in which the limiting magnetic field satisfies the stationary Euler equations, and more importantly retains the topology of the initial magnetic field $\boldsymbol{B}_0$.

While we established the global-in-time existence and uniqueness of weak solutions for the system (1.2) in Chapters 3 and 4, this was in the resistive ($\eta > 0$) case, and unfortunately the diffusion term in the $\boldsymbol{B}$ equation would destroy the topology of the initial magnetic field. In contrast, while both non-resistive systems (1.5) and (1.6) would preserve the topology of the magnetic field, the results in Chapters 5, 6 and 7 only yield local-in-time existence of solutions.

An essential first step to making the method of magnetic relaxation rigorous must thus consist of a global-in-time existence (and preferably uniqueness) proof of solutions to system (1.5). Of great interest is the recent paper of Brenier (2014), in which he proves the existence in 2D of a global dissipative solution to a related system of equations:

$$\begin{aligned} \boldsymbol{u} &= \Pi[(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}], \\ \frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{B} &= 0. \end{aligned}$$

It is clearly worth investigating whether these techniques could be adapted to systems (1.5) and (1.6).

Moreover, while Núñez (2007) proved that the kinetic energy of a smooth, bounded solution of the MHD system (1.5) must decay to zero, no such result is known for the Stokes-MHD system (1.6). Preliminary research (undertaken by the author as part of his MSc dissertation) suggested that additional hypotheses may be required to guarantee the same result, due to the absence of diffusion in the $\boldsymbol{u}$ equation; however, significant further work is needed to determine the efficacy of using system (1.6) to investigate magnetic relaxation.

Appendix A

An Alternative Commutator Estimate

In Chapter 7, we used Lemma 1.1 from Chemin (1992) (see Lemma 7.5) to prove one of the a priori estimates we needed for existence in Besov spaces. It is natural to ask whether it is possible to take all three norms on the right-hand side of Lemma 7.5 as homogeneous Sobolev norms, rather than inhomogeneous ones. In this appendix we prove that one can take all three norms to be homogeneous, but only provided that $s \geq 1$. (Unfortunately our application of Lemma 7.5 requires $s = n/2 - 1$, and thus the result in this appendix does not apply.)

Lemma A.1. *Let $s \geq 1$ and $s_1, s_2 > 0$ such that*

$$1 \leq s_1 < \tfrac{n}{2} + 1 \qquad \text{and} \qquad s_1 + s_2 = s + \tfrac{n}{2} + 1. \tag{A.1}$$

Then there exists a constant c such that for all $\boldsymbol{u}, \boldsymbol{B} \in \dot{H}^{s_1}(\mathbb{R}^n) \cap \dot{H}^{s_2}(\mathbb{R}^n)$ with $\nabla \cdot \boldsymbol{u} = 0$,

$$|\langle \Lambda^s[(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}], \Lambda^s \boldsymbol{B} \rangle| \leq c(\|\boldsymbol{u}\|_{\dot{H}^{s_1}} \|\boldsymbol{B}\|_{\dot{H}^{s_2}} + \|\boldsymbol{u}\|_{\dot{H}^{s_2}} \|\boldsymbol{B}\|_{\dot{H}^{s_1}}) \|\boldsymbol{B}\|_{\dot{H}^s}.$$

This is a partial generalisation of Lemma 1.1 from Chemin (1992). In fact, we prove the following slightly more general result.

Proposition A.2. *Let $s \geq 1$ and $s_1, s_2 > 0$ such that*

$$1 \leq s_1 < \tfrac{n}{2} + 1 \qquad \text{and} \qquad s_1 + s_2 = s + \tfrac{n}{2} + 1. \tag{A.1}$$

Then there exists a constant c such that for all $\boldsymbol{u}, \boldsymbol{B} \in \dot{H}^{s_1}(\mathbb{R}^n) \cap \dot{H}^{s_2}(\mathbb{R}^n)$,

$$\|\Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] - (\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B})\|_{L^2} \leq c(\|\boldsymbol{u}\|_{\dot{H}^{s_1}}\|\boldsymbol{B}\|_{\dot{H}^{s_2}} + \|\boldsymbol{u}\|_{\dot{H}^{s_2}}\|\boldsymbol{B}\|_{\dot{H}^{s_1}}).$$

Lemma A.1 follows immediately from Proposition A.2 using the fact that

$$\langle(\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B}), \Lambda^s\boldsymbol{B}\rangle = 0$$

whenever $\nabla\cdot\boldsymbol{u} = 0$. To prove Proposition A.2 we need a simple lemma.

Lemma A.3. *There exists a constant c such that, for any $i \in \mathbb{Z}$ and $1 \leq p \leq q \leq \infty$, if $\dot{\triangle}_i\boldsymbol{u} \in L^p(\mathbb{R}^n)$ then $\dot{\triangle}_i\boldsymbol{u} \in L^q(\mathbb{R}^n)$ and*

$$\|\dot{\triangle}_i\boldsymbol{u}\|_{L^q} \leq c2^{in(1/p-1/q)}\|\dot{\triangle}_i\boldsymbol{u}\|_{L^p}.$$

Proof. This follows immediately from Bernstein's inequality (3.8), since the support of $\widehat{\dot{\triangle}_i\boldsymbol{u}}$ is contained in a ball of radius 2^{i+1}. □

Proof of Proposition A.2. Write $\boldsymbol{u} = \sum_{i\in\mathbb{Z}}\dot{\triangle}_i\boldsymbol{u}$ and $\boldsymbol{B} = \sum_{j\in\mathbb{Z}}\dot{\triangle}_j\boldsymbol{B}$; then

$$\begin{aligned}
\boldsymbol{f} &= \Lambda^s[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] - (\boldsymbol{u}\cdot\nabla)(\Lambda^s\boldsymbol{B}) \\
&= \sum_{j\in\mathbb{Z}}\Lambda^s\left[\left(\sum_{i\in\mathbb{Z}}\dot{\triangle}_i\boldsymbol{u}\right)\nabla\dot{\triangle}_j\boldsymbol{B}\right] - \left(\sum_{i\in\mathbb{Z}}\dot{\triangle}_i\boldsymbol{u}\right)\nabla\Lambda^s\dot{\triangle}_j\boldsymbol{B} \\
&= \sum_{j\in\mathbb{Z}}\Lambda^s\left[\left(\sum_{i=-\infty}^{j-10}\dot{\triangle}_i\boldsymbol{u}\right)\nabla\dot{\triangle}_j\boldsymbol{B}\right] - \left(\sum_{i=-\infty}^{j-10}\dot{\triangle}_i\boldsymbol{u}\right)\nabla\Lambda^s\dot{\triangle}_j\boldsymbol{B} \\
&\qquad + \sum_{j\in\mathbb{Z}}\Lambda^s\left[\left(\sum_{i=j-9}^{j+9}\dot{\triangle}_i\boldsymbol{u}\right)\nabla\dot{\triangle}_j\boldsymbol{B}\right] - \left(\sum_{i=j-9}^{j+9}\dot{\triangle}_i\boldsymbol{u}\right)\nabla\Lambda^s\dot{\triangle}_j\boldsymbol{B} \\
&\qquad + \sum_{i\in\mathbb{Z}}\Lambda^s\left[\dot{\triangle}_i\boldsymbol{u}\left(\sum_{j=-\infty}^{i-10}\nabla\dot{\triangle}_j\boldsymbol{B}\right)\right] - \dot{\triangle}_i\boldsymbol{u}\left(\sum_{j=-\infty}^{i-10}\nabla\Lambda^s\dot{\triangle}_j\boldsymbol{B}\right) \\
&=: \sum_{j\in\mathbb{Z}}\boldsymbol{f}_{1,j} + \sum_{j\in\mathbb{Z}}\boldsymbol{f}_{2,j} + \sum_{i\in\mathbb{Z}}\boldsymbol{f}_{3,i}.
\end{aligned}$$

Taking the Fourier transform of $\boldsymbol{f}_{1,j}$, we have

$$\hat{\boldsymbol{f}}_{1,j}(\xi) = \int_{\mathbb{R}^n}(|\xi|^s - |\eta|^s)\sum_{i=-\infty}^{j-10}\widehat{\dot{\triangle}_i\boldsymbol{u}}(\xi-\eta)\eta\widehat{\dot{\triangle}_j\boldsymbol{B}}(\eta)\,\mathrm{d}\eta.$$

Since $i \leq j-10$, $|\xi-\eta| < |\eta|/2$, so by Lemma 5.2 we have

$$|\hat{\boldsymbol{f}}_{1,j}(\xi)| \leq \int_{\mathbb{R}^n} |\xi-\eta| \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i \boldsymbol{u}}(\xi-\eta) \right| |\eta|^s \widehat{\dot{\triangle}_j \boldsymbol{B}}(\eta)\, \mathrm{d}\eta.$$

Let q_1, q_2 satisfy $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{2}$ and $2<q_1<\frac{n}{s_1-1}$, and let p_1, p_2 satisfy $\frac{1}{p_i}=\frac{1}{q_i}+\frac{1}{2}$. Noting that $1+\frac{1}{2}=\frac{1}{p_1}+\frac{1}{p_2}$, by Young's inequality for convolutions we have

$$\|\hat{\boldsymbol{f}}_{1,j}\|_{L^2} \leq \left\| |\zeta| \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i \boldsymbol{u}}(\zeta) \right| \right\|_{L^{p_1}} \left\| |\eta|^s \widehat{\dot{\triangle}_j \boldsymbol{B}}(\eta) \right\|_{L^{p_2}}.$$

As $1-s_1+n/q_1>0$, by Hölder's inequality we have

$$\begin{aligned}
\left\| |\zeta| \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i \boldsymbol{u}}(\zeta) \right| \right\|_{L^{p_1}} &\leq \left\| |\zeta|^{1-s_1} \mathbb{1}_{\{|\zeta|\leq 2^{j-10}\}} \right\|_{L^{q_1}} \left\| |\zeta|^{s_1} \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i \boldsymbol{u}}(\zeta) \right| \right\|_{L^2} \\
&\leq c 2^{j(1-s_1+n/q_1)} \|\boldsymbol{u}\|_{\dot{H}^{s_1}}.
\end{aligned}$$

For the other term, by Hölder's inequality,

$$\begin{aligned}
\left\| |\eta|^s \widehat{\dot{\triangle}_j \boldsymbol{B}}(\eta) \right\|_{L^{p_2}} &\leq \left\| |\eta|^s \mathbb{1}_{\{2^{j-1}\leq|\zeta|\leq 2^{j+1}\}} \right\|_{L^{q_2}} \left\| \widehat{\dot{\triangle}_j \boldsymbol{B}}(\eta) \right\|_{L^2} \\
&\leq c 2^{j(s+n/q_2)} \left\| \dot{\triangle}_j \boldsymbol{B} \right\|_{L^2},
\end{aligned}$$

hence

$$\begin{aligned}
\|\boldsymbol{f}_{1,j}\|_{L^2} &\leq c \|\boldsymbol{u}\|_{\dot{H}^{s_1}} 2^{j(s-s_1+n/q_1+n/q_2+1)} \left\| \dot{\triangle}_j \boldsymbol{B} \right\|_{L^2} \\
&\leq c \|\boldsymbol{u}\|_{\dot{H}^{s_1}} 2^{js_2} \left\| \dot{\triangle}_j \boldsymbol{B} \right\|_{L^2}
\end{aligned}$$

and thus

$$\sum_{j\in\mathbb{Z}} \|\boldsymbol{f}_{1,j}\|_{L^2}^2 \leq c \|\boldsymbol{u}\|_{\dot{H}^{s_1}}^2 \|\boldsymbol{B}\|_{\dot{H}^{s_2}}^2. \tag{A.2}$$

For the second term, since $\left(\sum_{i=j-9}^{j+9} \dot{\triangle}_i \boldsymbol{u}\right) \nabla \dot{\triangle}_j \boldsymbol{B}$ is localised in Fourier space in an annulus centred at radius 2^j, we obtain

$$
\begin{aligned}
\|\boldsymbol{f}_{2,j}\|_{L^2} &\le \left\| \Lambda^s \left[\left(\sum_{i=j-9}^{j+9} \dot{\triangle}_i \boldsymbol{u} \right) \nabla \dot{\triangle}_j \boldsymbol{B} \right] \right\|_{L^2} + \left\| \left(\sum_{i=j-9}^{j+9} \dot{\triangle}_i \boldsymbol{u} \right) \nabla \Lambda^s \dot{\triangle}_j \boldsymbol{B} \right\|_{L^2} \\
&\le c 2^{js} \sum_{i=j-9}^{j+9} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^4} \|\nabla \dot{\triangle}_j \boldsymbol{B}\|_{L^4} + \sum_{i=j-9}^{j+9} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^4} \|\nabla \Lambda^s \dot{\triangle}_j \boldsymbol{B}\|_{L^4} \\
&\le c 2^{j(s+n/4)} \|\nabla \dot{\triangle}_j \boldsymbol{B}\|_{L^2} \sum_{i=j-9}^{j+9} 2^{in/4} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^2} \\
&\le c 2^{j(s+n/2-s_1)} \|\nabla \dot{\triangle}_j \boldsymbol{B}\|_{L^2} \sum_{i=j-9}^{j+9} 2^{j(s_1-n/4)} 2^{in/4} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^2}
\end{aligned}
$$

using Bernstein's inequality (Lemma A.3). Since $|i-j| \le 9$, $2^{j(s_1-n/4)} \le c 2^{i(s_1-n/4)}$, so

$$
\|\boldsymbol{f}_{2,j}\|_{L^2} \le c 2^{j(s_2-1)} \|\nabla \dot{\triangle}_j \boldsymbol{B}\|_{L^2} \sum_{i=j-9}^{j+9} 2^{is_1} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^2},
$$

and thus

$$
\sum_{j\in\mathbb{Z}} \|\boldsymbol{f}_{2,j}\|_{L^2}^2 \le c \, \|\boldsymbol{u}\|_{\dot{H}^{s_1}}^2 \, \|\boldsymbol{B}\|_{\dot{H}^{s_2}}^2 \,. \tag{A.3}
$$

For the third term, we use the Sobolev embedding

$$
\|\nabla \boldsymbol{u}\|_{L^p} \le c \|\boldsymbol{u}\|_{\dot{H}^{s_1}}
$$

provided $p = \frac{2n}{n-2s_1+2}$. Using Hölder's inequality, we obtain

$$
\begin{aligned}
\|\boldsymbol{f}_{3,i}\|_{L^2} &\le \left\| \Lambda^s \left[\dot{\triangle}_i \boldsymbol{u} \left(\sum_{j=-\infty}^{i-10} \nabla \dot{\triangle}_j \boldsymbol{B} \right) \right] \right\|_{L^2} + \left\| \dot{\triangle}_i \boldsymbol{u} \left(\sum_{j=-\infty}^{i-10} \nabla \Lambda^s \dot{\triangle}_j \boldsymbol{B} \right) \right\|_{L^2} \\
&\le 2^{is} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^{n/(s_1-1)}} \left\| \sum_{j=-\infty}^{i-10} \nabla \dot{\triangle}_j \boldsymbol{B} \right\|_{L^{2n/(n-2s_1+2)}} \\
&\quad + \|\dot{\triangle}_i \boldsymbol{u}\|_{L^{n/(s_1-1)}} \left\| \sum_{j=-\infty}^{i-10} \nabla \Lambda^s \dot{\triangle}_j \boldsymbol{B} \right\|_{L^{2n/(n-2s_1+2)}} \\
&\le c 2^{i(s+n/2+1-s_1)} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^2} \|\boldsymbol{B}\|_{\dot{H}^{s_1}} \\
&\le c 2^{is_2} \|\dot{\triangle}_i \boldsymbol{u}\|_{L^2} \|\boldsymbol{B}\|_{\dot{H}^{s_1}}
\end{aligned}
$$

using Bernstein's inequality (Lemma A.3) and the fact that $2^{js} \leq 2^{is}$. Hence

$$\sum_{i\in\mathbb{Z}} \|\boldsymbol{f}_{3,i}\|_{L^2}^2 \leq c\,\|\boldsymbol{u}\|_{\dot{H}^{s_2}}^2 \,\|\boldsymbol{B}\|_{\dot{H}^{s_1}}^2 . \tag{A.4}$$

Combining (A.2), (A.3) and (A.4) yields the desired result. □

In particular, taking $s = s_1 = n/2$ and $s_2 = n/2 + 1$ in Proposition A.2 yields

$$\begin{aligned}\|\Lambda^{n/2}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}] - (\boldsymbol{u}\cdot\nabla)(\Lambda^{n/2}\boldsymbol{B})\|_{L^2}& \\ \leq c(\|\nabla\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\boldsymbol{B}\|_{\dot{H}^{n/2}} &+ \|\boldsymbol{u}\|_{\dot{H}^{n/2}}\|\nabla\boldsymbol{B}\|_{\dot{H}^{n/2}}).\end{aligned}$$

The counterexample in Section 5.2 shows that one cannot remove the second term on the right-hand side, at least in the case $n = 2$. Even so, this result is useful in relation to the problem of lower bounds for potential blowup of solutions to the Navier–Stokes equations considered in Robinson, Sadowski & Silva (2012).

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